AFWAL-TR-83-2079

WEIBULL ANALYSIS HANDBOOK

Dr. R. B. Abernethy
J. E. Breneman
C. H. Medlin
G. L. Reinman

Pratt & Whitney Aircraft
Government Products Division
United Technologies Corporation
P.O. Box 2691
West Palm Beach, Florida 33402

November 1983

Final Report for Period 1 July 1982 to 31 August 1983

Aero Propulsion Laboratory
Air Force Wright Aeronautical Laboratories
Air Force Systems Command
Wright-Patterson AFB, Ohio 45433

NOTICE

This technical report has been reviewed and is approved for publication.

NANCY A. NOBLITT
Project Engineer

JACK RICHENS
Chief, Components Branch
Turbine Engine Division

FOR THE COMMANDER

H. I. BUSH
Director
Turbine Engine Division
Aero Propulsion Laboratory

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFWAL-TR-83-2079	2. GOVT ACCESSION NO. AD-A148100	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) WEIBULL ANALYSIS HANDBOOK		5. TYPE OF REPORT & PERIOD COVERED Final Report 1 July 1982 to 31 August 1983
		6. PERFORMING ORG. REPORT NUMBER P&WA/GPD FR-17579
7. AUTHOR(s) Dr. R. B. Abernethy; J. E. Breneman; C. H. Medlin; G. L. Reinman		8. CONTRACT OR GRANT NUMBER(s) F33615-82-C-2242
9. PERFORMING ORGANIZATION NAME AND ADDRESS United Technologies Corporation Pratt & Whitney Aircraft Group Government Products Division P.O. Box 2691, West Palm Beach, FL		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 30660649
11. CONTROLLING OFFICE NAME AND ADDRESS Aero Propulsion Laboratory (AFWAL/POTC) Air Force Wright Aeronautical Laboratories (AFSC) Wright Patterson Air Force Base, Ohio 45433		12. REPORT DATE November 1983
		13. NUMBER OF PAGES 228
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Weibull, Weibayes, Risk Analysis, Reliability Testing, Thorndike Charts, Fatigue Testing, Life Testing, Exponential Distribution

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This handbook is intended to provide instructions on how to do Weibull analysis. It will provide an understanding of Weibull analysis that is common between the military and industry. The handbook contains seven chapters plus an appendix. The chapters are written containing a minimum of mathematics with proofs given in the appendix.

FOREWORD

Advanced Weibull methods have been developed at Pratt & Whitney Aircraft in a joint effort between the Government Products Division and the Commercial Products Division. Although these methods have been used in aircraft engine projects in both Divisions, the advanced technologies have never been published, even though they have been presented and used by the U. S. Air Force (WPAFB), U. S. Navy (NAVAIR) and several component manufacturers.

The authors would like to acknowledge the contribution to this work made by several other Pratt & Whitney Aircraft employees: D. E. Andress, F. E. Dauser, J. W. Grdenick, J. H. Isiminger, B. J. Kracunas, R. Morin, M. E. Obernesser, M. A. Proschan, and B. G. Ringhiser.

The key Air Force personnel that encouraged publication were: Gary Adams, Dr. Tom Curran, Jim Day, Bill Troha and Don Zabierek (the USAF Program Manager), all at WPAFB.

The following members of the American Institute of Aeronautics and Astronautics Systems Effectiveness and Safety Committee provided valuable constructive reviews for which the authors are indebted: M. Berssenbrugge, R. Cosgrove, P. Dick, T. P. Enright, J. F. Kent, L. Knight, T. Prasinos, B. F. Shelley, and K. L. Wong.

The authors would be pleased to review constructive comments for future revisions.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Continued)

LIST OF ILLUSTRATIONS (Continued)

LIST OF ILLUSTRATIONS (Continued)

LIST OF ILLUSTRATIONS (Continued)

LIST OF TABLES

LIST OF TABLES (Continued)

CHAPTER 1

INTRODUCTION TO WEIBULL ANALYSIS

1.1 OBJECTIVE

The objective of this handbook is to provide an understanding of both the standard and advanced Weibull techniques that have been developed for failure analysis. The authors intend that their presentation be such that a novice engineer can perform Weibull analysis after studying this document.

1.2 BACKGROUND

Waloddi Weibull delivered his hallmark paper on this subject[1] in 1951. He claimed that his distribution, or more specifically his family of distributions, applied to a wide range of problems. He illustrated this point with seven examples ranging from the yield strength of steel to the size of adult males born in the British Isles. He claimed that the function "...may sometimes render good service". He did not claim that it always worked or even that it was always the best choice.

Time has shown that Waloddi Weibull was correct in all of these statements and particularly within the aerospace industry. The initial reaction to his paper in the 1950's and even the early 1960's was negative, varying from skepticism to outright rejection. Only after pioneers in the field experimented with the method and verified its wide application did it become popular. Today it has many applications in many industries and in particular the aerospace industry. There are special problems in aerospace and unusual arrays of data. Special methods had to be developed to apply the Weibull distribution. The authors believe there is a need for a standard reference for these newer methods as applied within the aerospace industry, and to industry in general.

1.3 EXAMPLES

The following are examples of aerospace problems that may be solved with Weibull analysis. It is the intent of this document to illustrate how to answer these and many similar questions through Weibull analysis.

- A project engineer reports three failures of his component in service operations in a six week period. Questions asked by the Program Manager are, "How many failures are predicted for the next three months, six months and one year?"
- "To order spare parts that may have a two to three year lead time, how may the number of engine modules that will be returned to a depot be forecast for three to five years hence month by month?"
- "What effect on maintainability support costs would the addition of the new split compressor case feature have relative to a full case?"
- "If the new Engineering Change eliminates an existing failure mode, how many units must be tested for how many hours without any failures to demonstrate with 90% confidence that the old failure mode has either been eliminated or significantly improved?"

[1] Weibull, Waloddi (1951). A Statistical Distribution Function of Wide Applicability. Journal of Applied Mechanics, pg. 293-297.

1.4 SCOPE

As treated herein, Weibull analysis application to failure analysis includes:

- Plotting the data
- Interpreting the plot
- Predicting future failures
- Evaluating various plans for corrective actions
- Substantiating engineering changes that correct failure modes.

Data problems and deficiencies are discussed with recommendations to overcome deficiencies such as:

- Censored data
- Mixtures of failure modes
- Nonzero time origin (t_o correction)
- No failures
- Extremely small samples
- Strengths and weaknesses of the method.

Statistical and mathematical derivations are presented in Appendices to supplement the main body of the handbook. There are brief discussions of alternative distributions such as the log normal. Actual case studies of aircraft engine problems are used for illustration. Where problems are presented for the reader to solve, answers are supplied. The use of Weibull distributions in mathematical models and simulations is also described.

1.5 ADVANTAGES OF WEIBULL ANALYSIS

One advantage of Weibull analysis is that it provides a simple graphical solution. The process consists of plotting a curve and analyzing it. (Figure 1.1). The horizontal scale is some measure of life, perhaps start/stop cycles, operating time, or gas turbine engine mission cycles. The vertical scale is the probability of the occurrence of the event. The slope of the line (β) is particularly significant and may provide a clue to the physics of the failure in question. The relationship between various values of the slope and typical failure modes is shown in Figure 1.2. This type of analysis relating the slope to possible failure modes can be expanded by inspecting libraries of past Weibull curves.

Another advantage of Weibull analysis is that it may be useful even with inadequacies in the data, as will be indicated later in the section. For example, the technique works with small samples. Methods will be described for identifying mixtures of failures, classes or modes, problems with the origin being at other than zero time, investigations of alternative scales other than time, non-serialized parts and components where the time on the part cannot be clearly identified, and even the construction of a Weibull curve when there are no failures at all, only success data.

In addition, as there are only a few alternatives to the Weibull, it is not difficult to make graphic comparisons to determine which distribution best fits the data. Further, if there is engineering evidence supporting another distribution, this should be considered and weighted heavily against the Weibull. However, it has been the writers' experience that the Weibull distribution most frequently provides the best fit of the type of data experienced in the gas turbine industry.

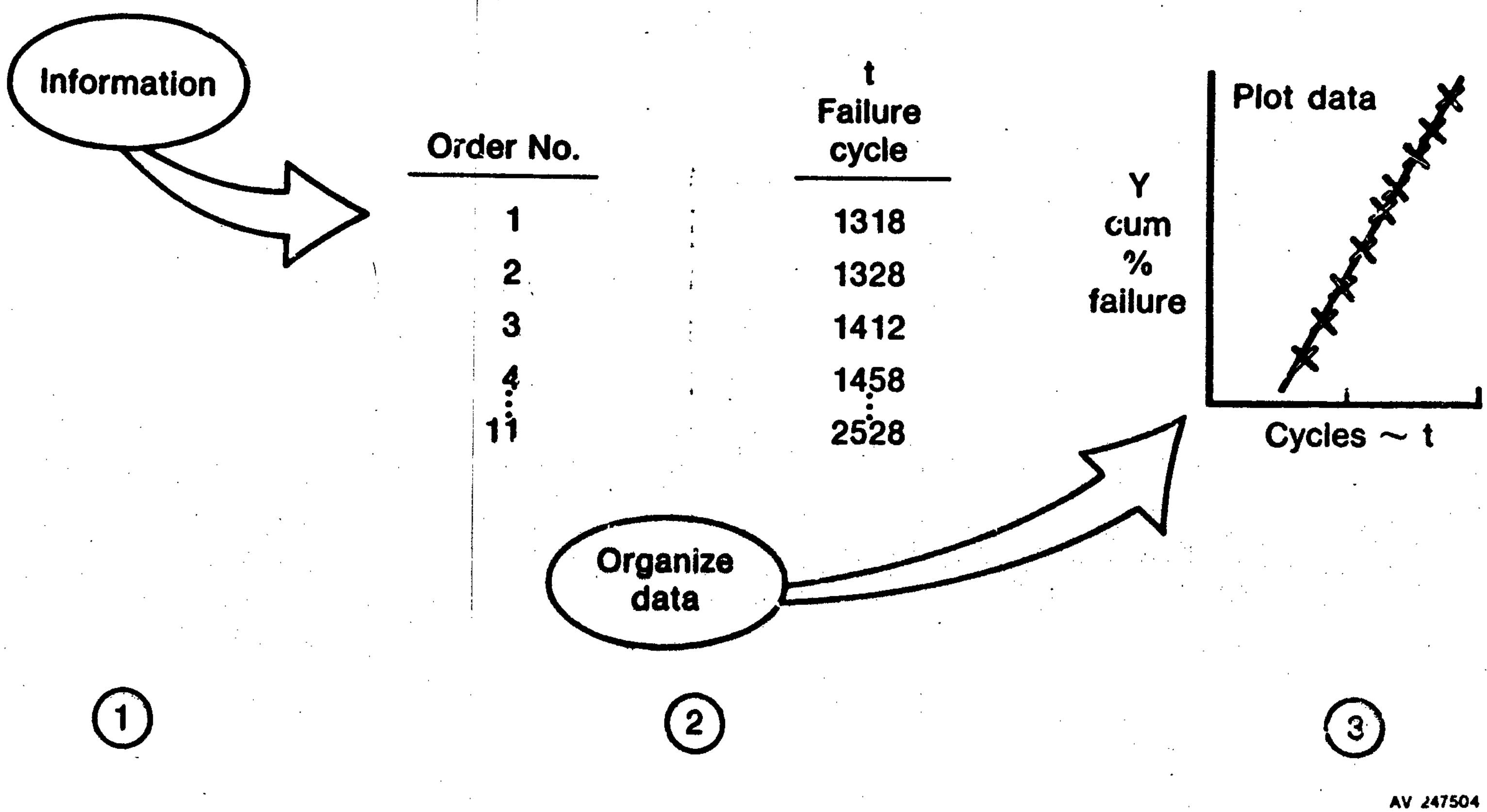


Figure 1.1. Performing a Weibull Analysis

1. Infant mortality
   - Inadequate burn-in, green run
   - Misassembly
   - Some quality problems
2. Random failures
   - Independent of time
   - Maintenance errors
   - Electronics
   - Mixtures of problems
3. Early wearout
   - Surprise!
   - Low cycle fatigue
4. Old age wearout (rapid)
   - Bearings
   - Corrosion

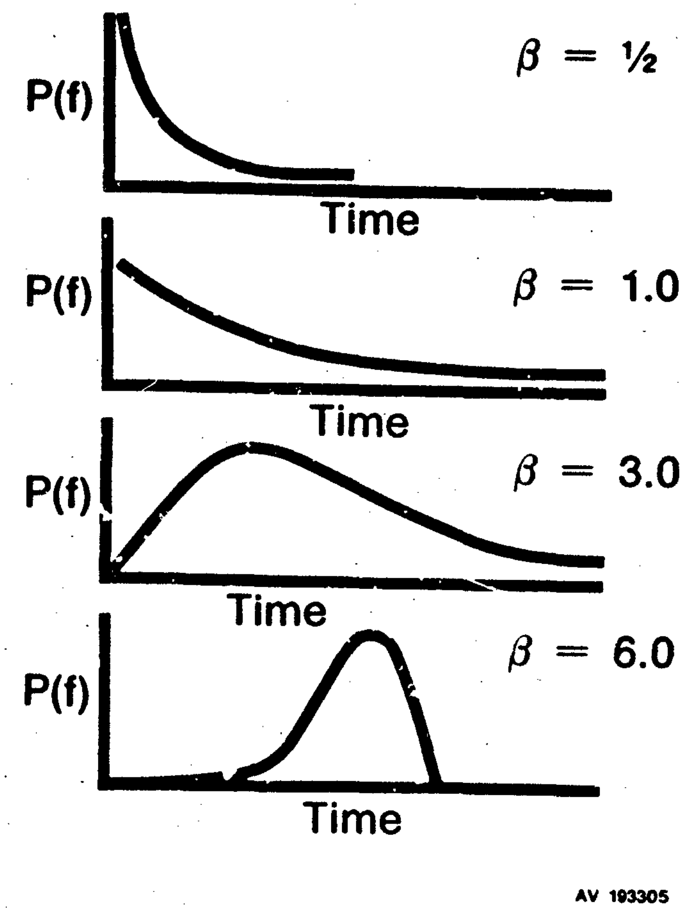


Figure 1.2. Understanding the Results

1.6 AGING TIME OR CYCLES

Most applications of Weibull analysis are based on a single failure class or mode from a single part or component. An ideal application would consist of a sample of 20 to 30 failures. Except for material characterization laboratory tests, ideal data are rare; usually the analysis is started with a few failures embedded in a large number of successful, unfailed or censored units. The age of each part is required. The units of age depend on the part usage and the failure mode. For example, low and high cycle fatigue may produce cracks leading to rupture. The age units would be fatigue cycles. The age unit of a jet starter may be the number of engine starts. Burner and turbine parts may fail as a function of time at high temperature or as the number of excursions from cold to hot and return. In most cases, knowledge of the physics-of-failure will provide the age scale When the units of age are unknown, several age scales must be tried to determine the best fit.

1.7 FAILURE DISTRIBUTION

The first use of the Weibull plot will be to determine the parameter β, which is known as the slope, or shape parameter. Beta determines which member of the family of Weibull failure distributions best fits or describes the data. The failure mode may be any one of the types represented by the familiar reliability bathtub curve, infant mortality with slopes less than one, random with slopes of one, and wearout with slopes greater than one. See Figure 1.2. The Weibull plot is also inspected to determine the onset of the failure. For example, it may be of interest to determine the time at which 1% of the population will have failed. This is called B1 life. Alternatively, it may be of interest in determining the time at which one tenth of 1% of the population will have failed, which is called B.1 life. These values can be read from the curve by inspection. See Figure 1.3.

1.8 RISK PREDICTIONS

If the failure occurred in service operations, the responsible engineer will be interested in a prediction of the number of failures that might be expected over the next three months, six months, a year, or two years. Methods for making these predictions are treated in Chapter 3. A typical risk prediction is shown in Table 1.1. This process may provide information on whether or not the failure mode applies to the entire fleet or to only one portion of the fleet, which is often called a batch. After the responsible engineer develops alternative plans for corrective action, including production rates and retrofit dates, the risk predictions will be repeated. The decision maker will require these risk predictions in order to select the best course of action.

1.9 ENGINEERING CHANGES AND MAINTENANCE PLAN EVALUATION

Weibull analysis is used to evaluate engineering changes as to their effect on the entire fleet of engines. Maintenance schedules and plans are also evaluated using Weibull analysis. These techniques are illustrated in Chapter 6 — Case Histories with Weibull Applications. In each case the baseline Weibull analysis is conducted without the engineering change or maintenance change. The study is then repeated with the estimated effect of the change modifying the Weibull curve. The difference in the two risk predictions represents the net effect of the change. The risk parameters may be the predicted number of failures, life cycle cost, depot loading, spare parts usage, hazard rate, or aircraft availability.

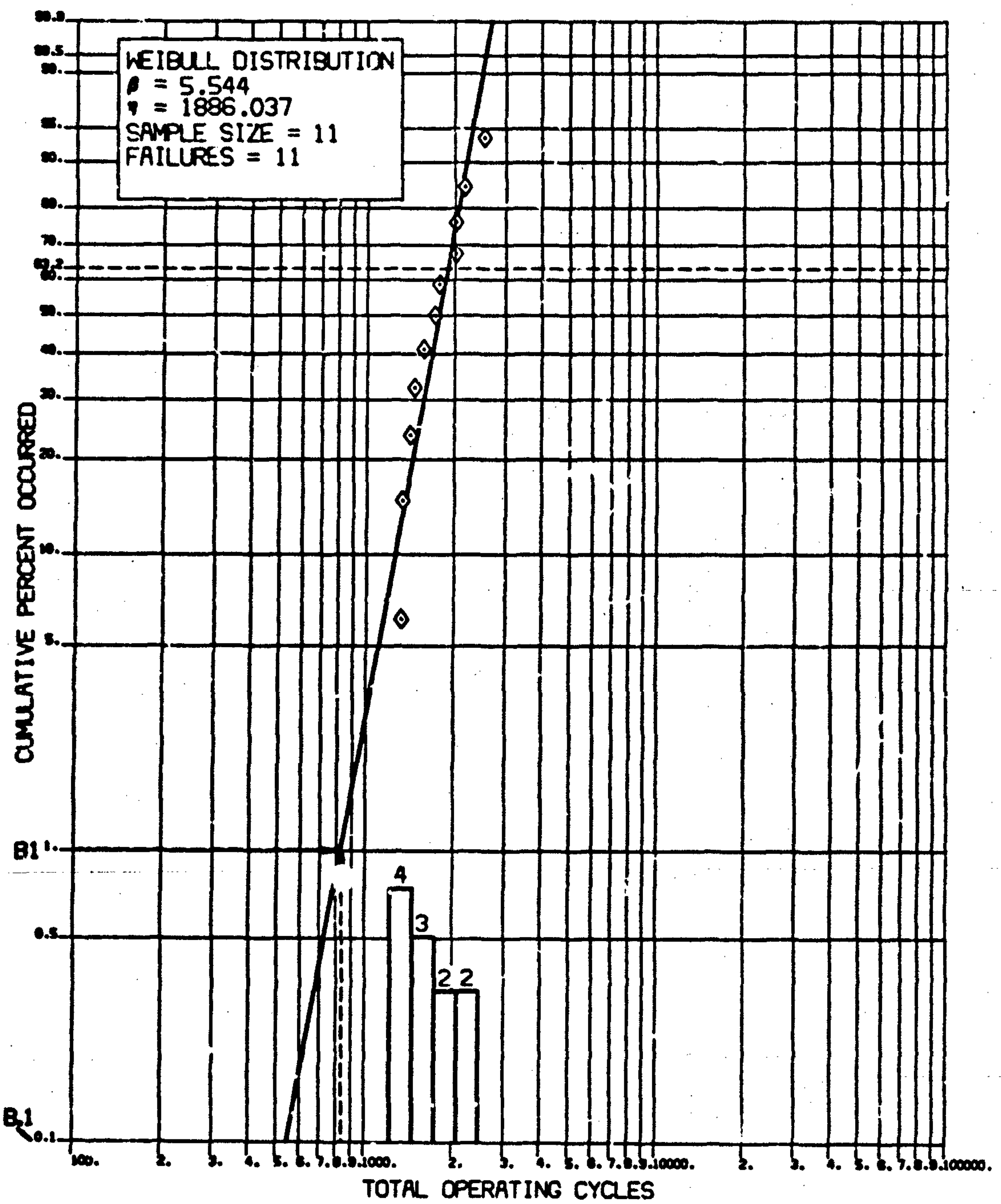


Figure 1.3. ID Louver Cracking

TABLE 1.1. WEIBULL RISK FORECAST

Risk Prediction for 12 Months Beginning July 1978

11.77	0.00 more failures in	0 months
15.12	3.35 more failures in	1 month
19.18	7.41 more failures in	2 months
24.07	12.30 more failures in	3 months
29.87	18.10 more failures in	4 months
36.69	24.92 more failures in	5 months
44.60	32.33 more failures in	6 months
53.68	41.91 more failures in	7 months
63.97	52.20 more failures in	8 months
75.53	63.76 more failures in	9 months
88.35	76.58 more failures in	10 months
102.42	90.65 more failures in	11 months
117.69	105.92 more failures in	12 months

What if? — Corrective action next month, next year

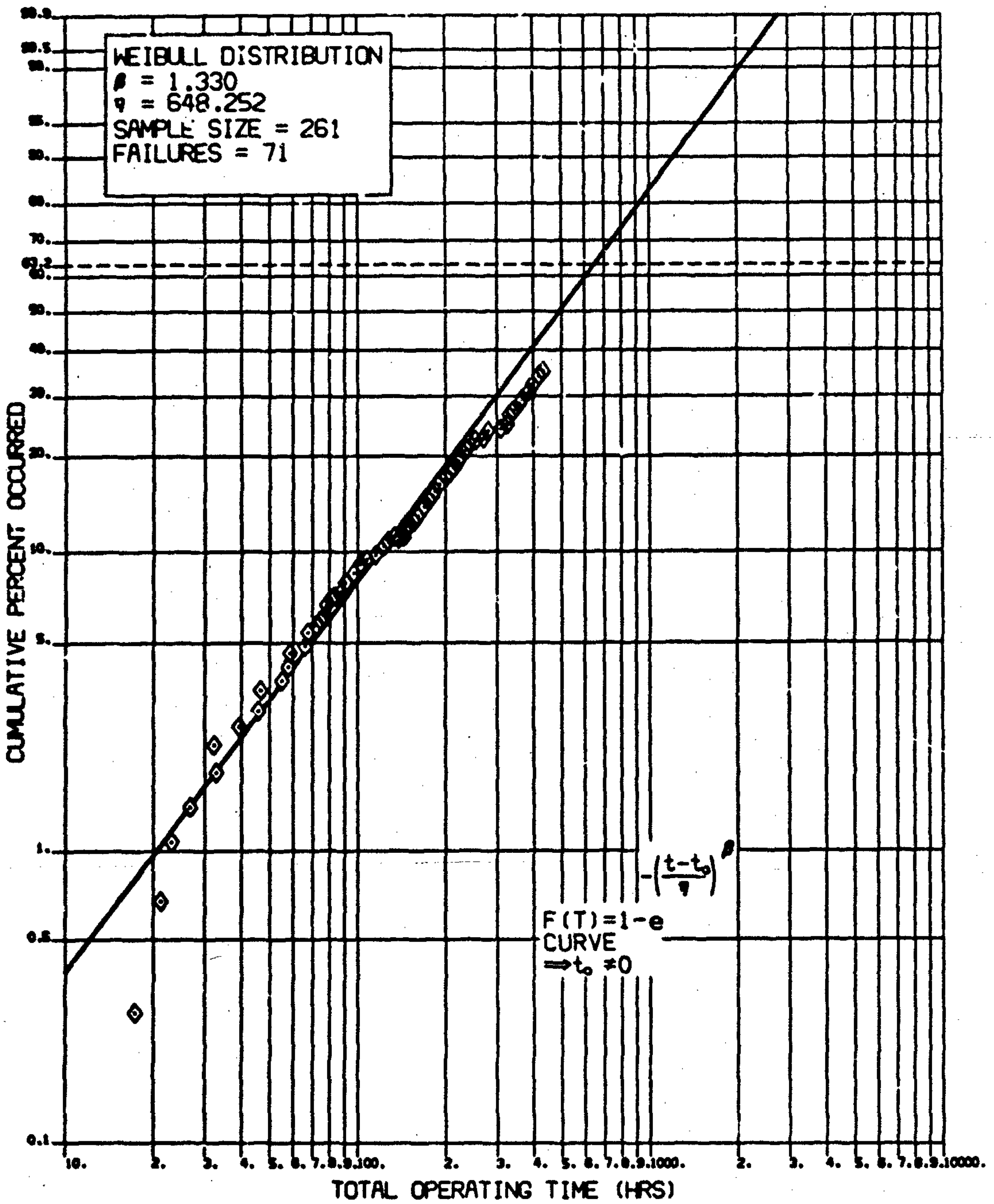
1.10 MATHEMATICAL MODELS

Mathematical models of an entire engine system including its control system may be produced by combining the effects of several hundred failure modes. The combination may be done by Monte Carlo simulation or by analytical methods. These models have been useful for predicting spare parts usage, availability, module returns to depot, and maintainability support costs. Generally, these models are updated with the latest Weibulls once or twice a year and predictions are regenerated for review.

1.11 WEIBULLS WITH CUSPS OR CURVES

The Weibull plot should be inspected to determine how well the failure data fit the straight line. The scatter should be evenly distributed about the line. However, sometimes the failure points will not fall on a straight line on the Weibull plot, and modification of the simple Weibull approach may be required. The bad fit may relate to the physics of the failure or to the quality of the data. There are at least two reasons why a bad fit may occur. First, the origin — If the points fall on gentle curves, it may be that the origin of the age scale is not located at zero See Figure 1.4. There may be physical reasons why this will be true. For example, with roller bearing unbalance, it may take a minimum amount of time for the wobbling roller to destroy the cage. This would lead to an origin correction equal to the minimum time. The origin correction may be either positive or negative. A procedure for determining the origin correction is given in Chapter 2.

Second, a mixture of failure modes — Sometimes the plot of the failure points will show cusps in sharp corners. This is an indication that there is more than one failure mode, i.e. a mixture of failure modes. See Figure 1.5. In this case it is necessary to conduct a laboratory failure analysis of each failure to determine if separate failure modes are present. If this is found to be the case, then separate Weibull plots are made for each set of data for each failure mode. If the laboratory analysis successfully categorized the failures into separate failure modes, the separate Weibull plots will show straight line fits, that is, very little data scatter. On each plot the failure data points from the other failure modes are treated as successful (censored or non-failure) units.



FD 272260

Figure 1.4. T_o Correction to Curved Weibull

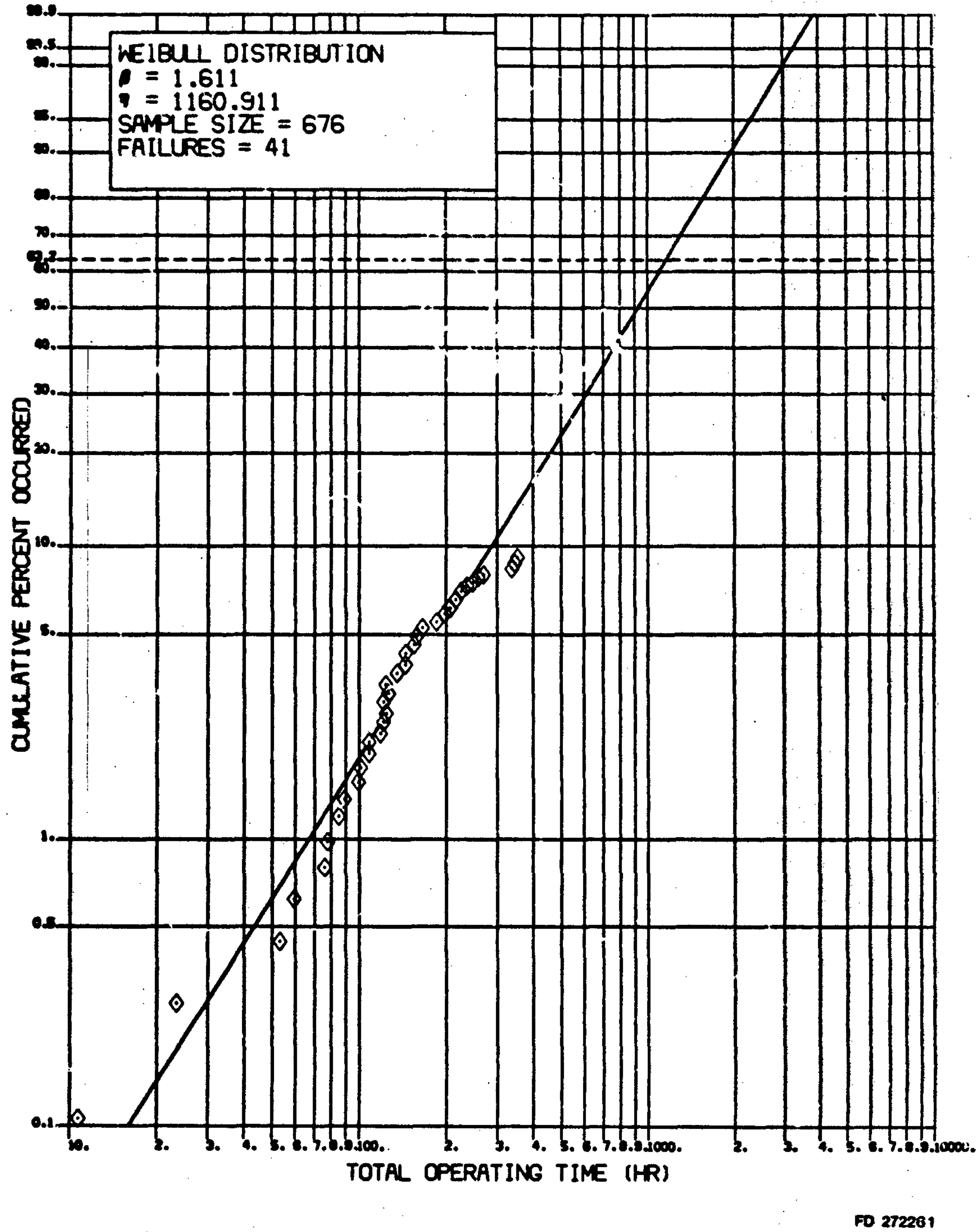


Figure 1.5. Mixing Failure Modes, MFP

1.12 SYSTEM WEIBULLS

If the data from a system such as a jet engine are not adequate to plot individual failure modes, it is tempting to plot a single Weibull for the system based on mean-time-between-failures (MFBF), assuming $\beta = 1$. This approach is fraught with difficulties and should be avoided if possible. However, there may be no alternative if the system does not have serialized part identification or the data do not identify the type of failure for each failure time. Some years ago it was popular to produce system Weibulls for the useful life period (Figure 1.6) assuming constant failure rate ($\beta = 1.0$). Electronic systems that do not have wearout modes were often analyzed in this manner. More recently, some studies indicate electronics may have a decreasing failure rate, i.e. a β of less than one.[1] Although data deficiencies may force the use of system Weibull analysis, a math model combining individual Weibull modes is preferred because it will be more useful and accurate.

1.13 NO-FAILURE WEIBULLS

In some cases, there is a need for a Weibull plot even when no failures have occurred. For example, if an engineering change or a maintenance plan modification is made to correct a failure mode experienced in service, how much success time is required before it can be stated (with some level of confidence) that the problem has been corrected. When parts approach or exceed their predicted design life, it may be possible to extend their predicted life by constructing a Weibull for evaluation even though no failures have occurred. A method called Weibayes analysis has been developed for this purpose and is presented in Chapter 4. Methods to design experiments to substantiate new designs using Weibayes theory are presented in Chapter 5 — Substantiation and Reliability Testing.

1.14 SMALL FAILURE SAMPLE WEIBULLS

Flight safety considerations may require using samples as small as two or three units. Weibull analysis, like any statistical analysis, is less precise with small samples. To evaluate these small-sample problems, extensive Monte Carlo and analytical studies have been made and will be presented in Appendix F. In general, small sample estimates of β tend to be too high (or steep) and the characteristic life, η, tends to be low. See Figure 1.7.

1.15 CHANGING WEIBULLS

After the initial Weibull plot is made, later plots will be based on larger failure samples and more time on successful units. Each plot will be slightly different, but gradually the Weibull parameters will stabilize as the data sample increases. The important inferences about B.1 life and the risk predictions are that they should not change significantly with a moderate size sample.

[1] "Unified Field (Failure) Theory-Demise of the Bathtub Curve", Kam LiWong, 1981 Proceedings Annual Reliability and Maintainability Symposium.

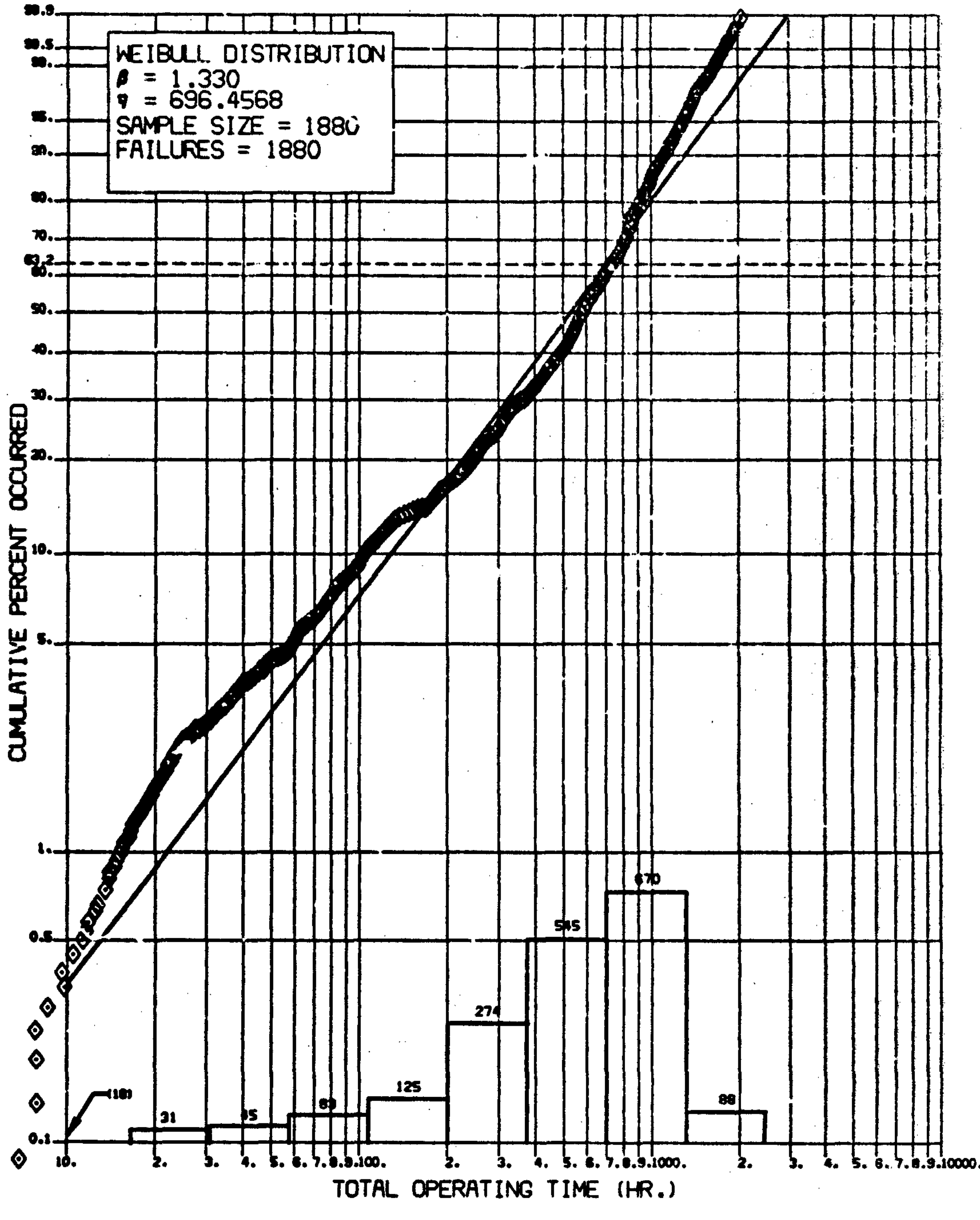


Figure 1.6. Mixing Failure Modes, EEC

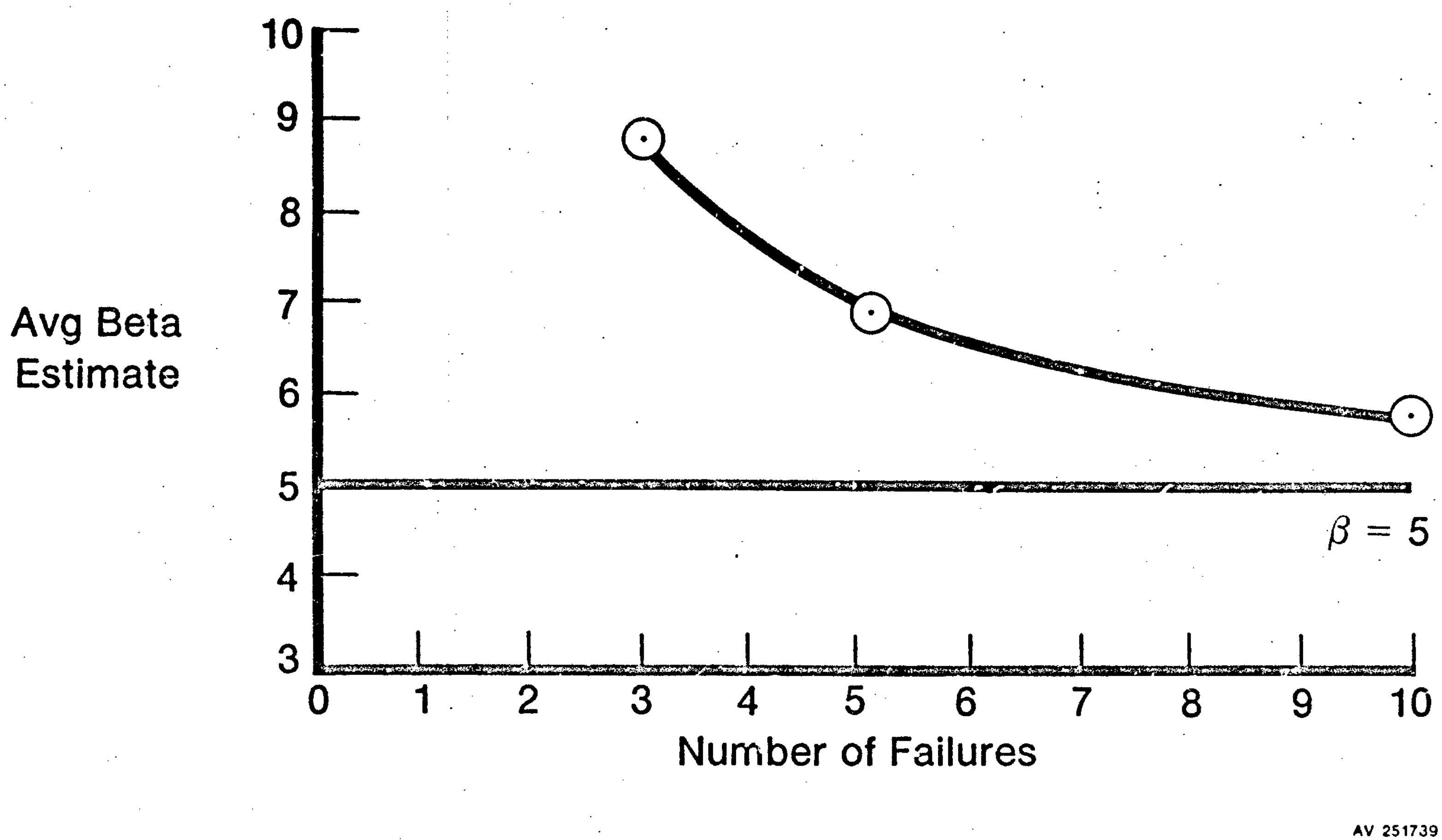


Figure 1.7. Small Sample Beta Estimates Are Too Steep

1.16 ESTABLISHING THE WEIBULL LINE

The standard approach for constructing Weibull plots is to plot the time-to-failure data on Weibull probability graphs using median rank plotting positions as described in Chapter 2. A straight line is then fit to the data to obtain estimates of β and η. This approach has some deficiencies as noted above for small samples but is simple and graphical. Maximum likelihood estimates may be more accurate, but require complex computer routines. The advantages and disadvantages of these methods are discussed in Appendices C and D.

1.17 SUMMARY

The authors' intent is that the material in this handbook will provide an understanding of this valuable tool for aerospace engineers in industry and Government. Constructive comments would be appreciated for future revisions of this handbook.

CHAPTER 2

PERFORMING A WEIBULL ANALYSIS

2.1 FOREWORD

This section describes how to construct Weibull paper and how to plot the data. Since interpretation of the data is the most important part of doing an analysis, an extensive discussion is given on how to interpret a Weibull plot. Examples are used to illustrate interpretation problems.

The first question to be answered is whether or not the data can be described by a Weibull distribution. If the data plots on a straight line on Weibull paper, the data can be approximated by a Weibull distribution.

2.2 WEIBULL PAPER AND ITS CONSTRUCTION

The Weibull distribution may be defined mathematically as follows:

$$F(t) = 1 - e^{-((t - t_o)/\eta)^\beta}$$

where:

$F(t)$	=	fraction failing
t	=	failure time
t_o	=	starting point or origin of the distribution
η	=	characteristic life or scale parameter
β	=	slope or shape parameter
e	=	exponential.

F(t) thus defines the cumulative fraction of a group of parts which will fail by a time t. Therefore, the fraction of parts which have not failed up to time t is 1 — F(t). This is often called reliability at time t and is denoted by R(t). By rearranging the distribution function, the following can be noted:

$$1 - F(t) = e^{-((t - t_o)/\eta)^\beta}$$

$$\text{let } t_o = 0$$

then

$$1 - F(t) = e^{-(t/\eta)^\beta}$$

$$\frac{1}{1 - F(t)} = e^{(t/\eta)^{\beta}}$$

$$\ln\left(\frac{1}{1 - F(t)}\right) = \left(\frac{t}{\eta}\right)^{\beta}$$

$$\ln \ln\left(\frac{1}{1 - F(t)}\right) = \beta \ln t - \beta \ln \eta$$

$$Y = BX + A$$

The expression Y = BX + A is the familiar equation for a straight line. By choosing $\ln t$ as X, the scale on the abscissa, and

$$\ln \ln\left(\frac{1}{1 - F(t)}\right)$$

as y, the scale on the ordinate, the cumulative Weibull distribution can be represented as a straight line. As noted in Tables 2.1, 2.2, and Figure 2.1, Weibull paper can be constructed as follows:

TABLE 2.1. CONSTRUCTION OF ORDINATE (Y)

F(t)	$\ln \ln \frac{1}{1 - F(t)}$	*Col 2 Value — Min Col 2 Value (−6.91)*
0.001	−6.91	0 units
0.01	−4.60	2.31
0.1	−2.25	4.66
0.5	−0.37	6.54
0.9	0.83	7.74
0.99	1.53	8.44
0.999	1.93	8.84

TABLE 2.2. CONSTRUCTION OF ABSCISSA (t)

t (hr)	*ln(t)*
1	0 units
2	0.69
3	1.10
4	1.39
5	1.61
10	2.30
15	2.71
20	3.00
.	.
.	.
.	.
100	4.61
1000	6.91

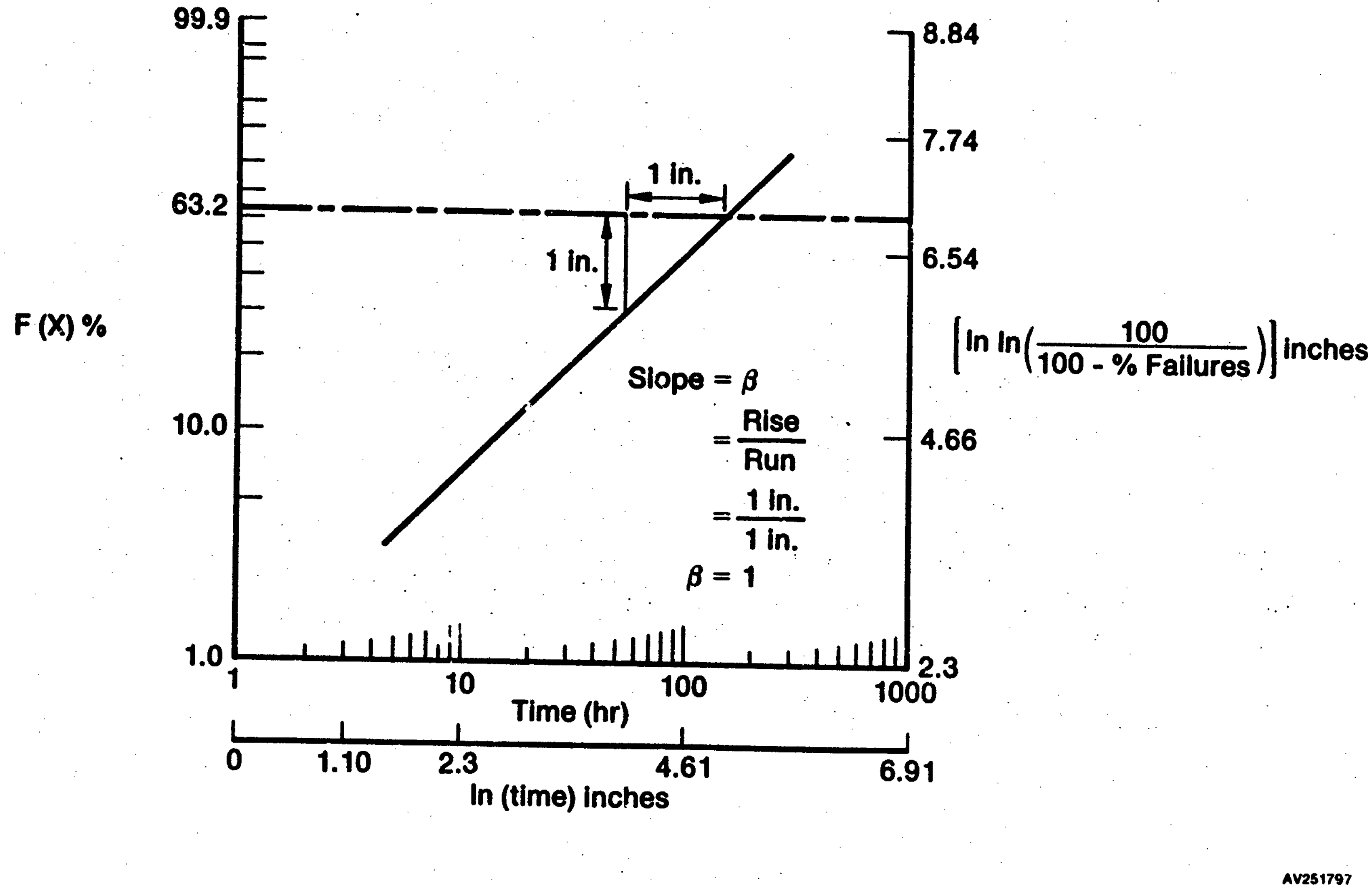


Figure 2.1. Construction of Weibull Paper

If the units used are common for the abscissa and the ordinate (i.e., inches to inches or centimeters to centimeters), the paper will have a one-to-one relationship for establishing the slope of the Weibull. (The Weibull parameter β is established by simply measuring the slope of the line on Weibull paper.) Of course, the scales can be made in any relationship. That is 2-to-1, 10-to-1, 100-to-1, or any other combination to best depict the data. Throughout this handbook data has been plotted on 1-to-1 paper wherever possible. However, the slopes will be displayed on the charts. Sample Weibull paper has been included in Appendix L. (At first glance, this paper may appear to be common log or log-log paper. Looks are deceiving because it is not and should not be used as such; nor can common log paper be used as Weibull paper.)

2.3 FAILURE DATA ANALYSIS — EXAMPLE

During the development, testing, and field operation of gas turbine engines, items sometimes fail. If the failure does not affect the performance of the aircraft, it will go unnoticed until the engine is removed and inspected. This was the case for the compressor inlet airseal rivets in the following example. The flare part of the rivet was found missing from one or more of the rivets during inspection.

A program was put into operation to replace the rivets with rivets of a new design. A fatigue comparison was to be used to verify the improvement in the new rivet. A baseline using the old rivets was established by an accelerated laboratory test. The results are presented in Table 2.3.

TABLE 2.3. BASELINE

Rivet Serial Number (S/N)	*Failure Time (min)*	*Remarks*
1	90	Failure
2	96	Failure
3	100	Rivet flare loosened without failure
4	30	Failure
5	49	Failure
6	45	Rivet flare loosened without failure
7	10	Lug failed at rivet attachment
8	82	Failure

Since rivet numbers 3, 6, and 7 were considered nonrepresentative failures, these data will be ignored for the first analysis. That leaves five data points. The first step in establishing a Weibull plot is to order the data from low time to high time failure. This facilitates establishing the plotting positions on the time axis. It is also needed to establish the corresponding ordinate F(t) values. Each failure in a group of tested units will have a certain percentage of the total population failing before it. These true values are seldom known. Studies[1] have been made as to how best to account for this inaccuracy, especially with small samples. However, most of these studies are limited, and more detailed discussion is beyond the scope of this handbook. It has been the convention at P&WA to use "Median Ranks" for establishing F(t) plotting positions, and tables can be found in Appendix B.

With five failures, the column in Appendix B headed with sample size 5 is used. The resulting coordinates for plotting the Weibull are shown in Table 2.4 and plotted in Figure 2.2. One additional item should be noted. Points with the same time should be plotted at that time at separate median rank values.

[1]Kapur and Lamberson, Reliability in Engineering Design, Wiley, pp 297-303.

TABLE 2.4. WEIBULL COORDINATES

Order Number	*S/N*	*Failure Time (Min)*	*Median Rank*
1	4	30	12.9
2	5	49	31.3
3	8	82	50.0
4	1	90	68.6
5	2	96	87.0

A line is drawn through the data points. Formal methods of rank regression and maximum likelihood for establishing the line are discussed in Appendices C and D respectively. The slope of the line is measured by taking the ratio of rise over run. Select a starting point and measure one inch in the horizontal direction (run). Then, measure vertically (rise), until the line is intersected. In Figure 2.2, the rise is two inches. Therefore, the slope represented by Greek symbol β, (β) = rise/run = 2/1 = 2. One needs two parameters to describe a Weibull distribution when discussing or reproducing the curve. The first is β, and the other is the characteristic life eta (denoted by η). Eta occurs at the 63.2 percentile of the distribution and is indicated on most Weibull paper. In Figure 2.2, the 63.2 percentile crosses the line at 80 minutes; therefore, the characteristic life η = 80 min.

The unique feature of the characteristic life is that it occurs at the 63.2% point regardless of the Weibull distribution (i.e., slope). By examining the Weibull equation it will become clear why this is true. When time, t, is equal to η it does not matter what β is; F(t) is always 63.2%:

$$
\begin{aligned}
F(t) &= 1 - e^{-(t/\eta)^{\beta}} \\
&= 1 - e^{-(1)^{\beta}} \text{ when } t = \eta \\
&= 1 - 0.368 \\
F(t) &= 0.632 \text{ regardless of the value of } \beta
\end{aligned}
$$

2.4 SUSPENDED TEST ITEMS — NONFAILURES

In the example in Section 2.3, some rivets failed by causes other than the failure mode of interest. A rivet that failed by a different mode cannot be plotted on the same Weibull chart in the same manner as a rivet which fractured because the rivets do not belong to the same failure distribution. These data points are referred to as suspended or censored points. There are several definitions[1] of suspensions, but for Weibull analysis, they are always treated the same way. They cannot be ignored when establishing the Weibull. The argument for including them in the analysis is that if their failure had occurred in the same fashion as other failures, the rank order of the other failures would have been influenced. Therefore, something needs to be done to account for the potential influence of these points. To illustrate the adjustment of the rank order numbers for the influence of these suspended items, the rivet test results will be used again.

[1]Type I: Test terminated after a fixed time has elapsed.
Type II: Test terminated after a set number of failures have occurred.
Type III: Test terminated for a cause other than the one of interest.

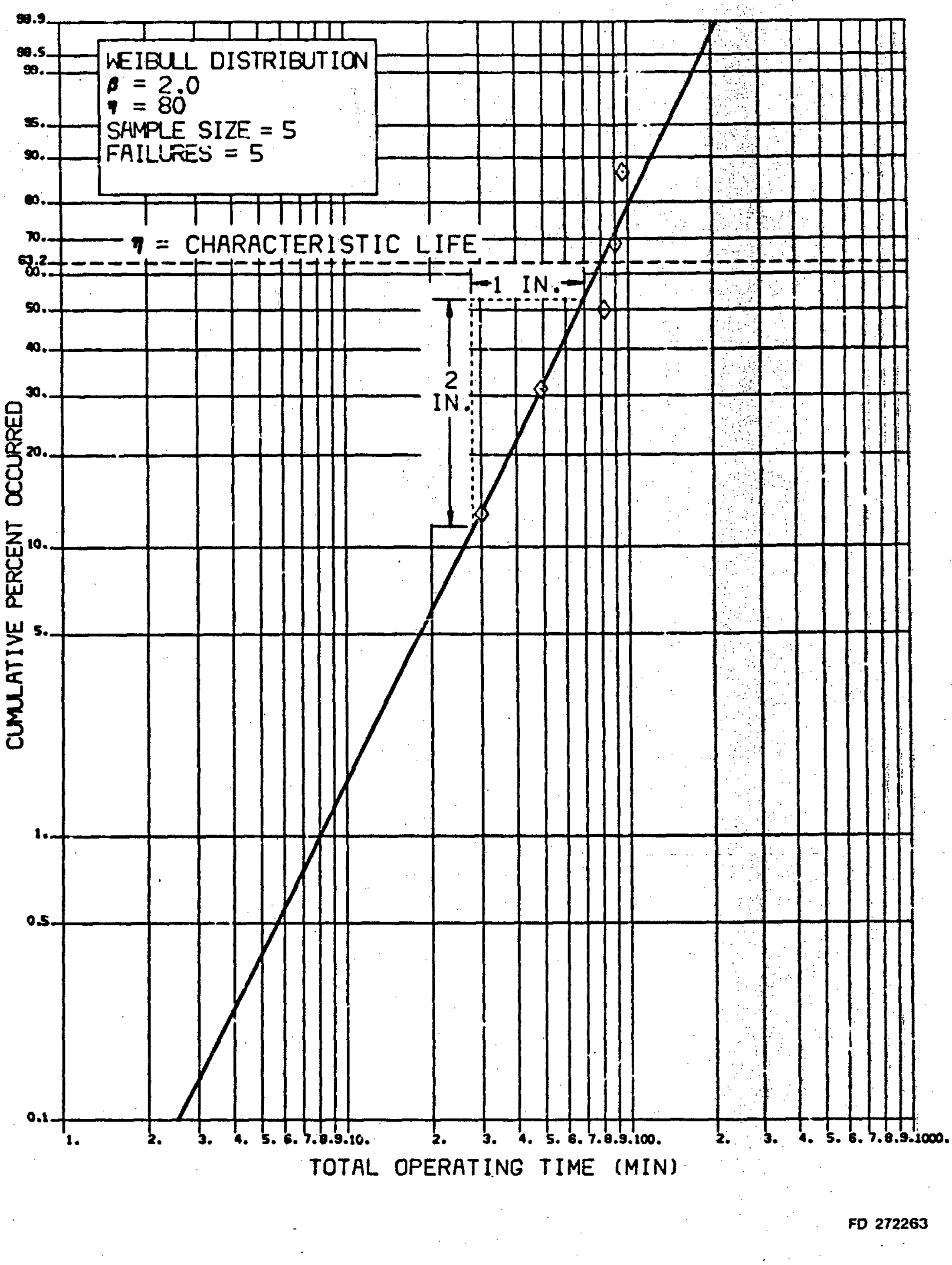


Figure 2.2. Rivet Failures

The general formula for adjusting the rank position, considering all possible ways the suspended item may have failed and potentially influenced the results, is given by the following equation:

$$\text{Rank Increment}^2 = \frac{(N + 1) - (\text{previous adjusted rank})}{1 + (\text{number of items beyond present suspended item})} \quad (2.2)$$

where N is the total number of rivets tested regardless of whether it failed, was suspended (Type I or Type II), or suspended by the wrong failure mode (Type III).

Applying this equation to the rivet test data, the values in Table 2.5 are obtained.

TABLE 2.5. ADJUSTED RANK

Rivet S/N	*Order*	*Time (minutes)*	*Adjusted Rank*
7	1	10 suspension	—
4	2	30 failure	1.125
6	3	45 suspension	—
5	4	49 failure	2.438
8	5	82 failure	3.751
1	6	90 failure	5.064
2	7	96 failure	6.377
3	8	100 suspension	—

The adjusted ranks were calculated in the following manner:

- Rivet No. 7 is a suspension; therefore, it does not need a rank value because it will not be plotted on the Weibull chart.

$$\text{Rank Increment for Rivet No. 4} = \frac{(8 + 1) - 0}{1 + 7} = 1.125$$

where:

8 is the total number of rivets tested whether they failed or not

0 is the previous adjusted rank (in this case there was none)

7 is the total number of items beyond the first suspension starting the count with the first failure as illustrated below:

Rivet	*Time*	*Items Beyond Suspension*
7	10 suspension	
4	30 failure	1 Starting here and counting forward
6	45 suspension	2
5	49 failure	3
8	82 failure	4
1	90 failure	5
2	96 failure	6
3	100 suspension	7

[2] Johnson, Leonard G. (1959). The Statistical Treatment of Fatigue Experiments. Research Laboratories, General Motors Corporation, pp. 44-50.

The adjusted rank is the previous rank (in this case 0) plus the rank increment of 1.125. Therefore, the adjusted rank is:

Adjusted Rank for Rivet No. 4 = 0 + 1.125 = 1.125

- Rivet No. 6 is a suspension and receives no rank value.

- Rivet No. 5 is a failure and the formula has to be employed again to identify the new rank increment to use between failures.

$$\text{Rank Increment} = \frac{(8 + 1) - 1.125}{1 + 5} = 1.313$$

where

1.125 is the previous adjusted rank

5 is the number of items beyond the last suspension starting with the tailure following that suspension.

Rivet	*Time*	*Items Beyond Suspension*
7	10 suspension	
4	40 failure	
6	45 suspension	
5	49 failure	1 Starting here and counting forward
8	82 failure	2
1	90 failure	3
2	96 failure	4
3	100 suspension	5

The adjusted rank, therefore, is the previous adjusted rank plus the new rank increment.

Adjusted Rank No. 5 = 1.125 + 1.313 = 2.438

- Rivets No. 5, Nc. 8, No. 1 and 2 are failures without any additional suspensions between them and the previous failures. Therefore, no new rank increment needs to be calculated. The last value calculated (1.313) is still valid. Therefore, the adjusted ranks for these rivets are:

  - Adjusted Rank No. 8 = 2.438 + 1.313 = 3.751
  - Adjusted Rank No. 1 = 3.751 + 1.313 = 5.064
  - Adjusted Rank No. 2 = 5.064 + 1.313 = 6.377.

With these adjusted ranks, the median ranks can be established. The sample size to be used when entering the median rank table would be 8. While interpolation could be used for determining the appropriate median rank, a good approximation is provided by Benard's formula[1]:

$$P_{0.50} = \frac{i - 0.3}{N + 0.4} \times 100\% \tag{2.4}$$

where N = sample size

i = adjusted rank value

Use of this formula is illustrated in Table 2.6.

$$P_{0.50} = \frac{(1.125 - 0.3) \times 100\%}{8 + 0.4} = 9.82\%$$

$$P_{0.50} = \frac{(2.438 - 0.3) \times 100\%}{8 + 0.4} = 25.45\%$$

etc.

TABLE 2.6. MEDIAN RANK

Adjusted Rank Order No.	*Median Rank*
1.125	9.82%
2.438	25.45%
3.751	41.08%
5.064	56.71%
6.377	72.35%

Using the calculated median ranks and the failure times, Figure 2.3 is derived. The slope of the line, $\beta = 2.0$, is the same as the earlier Weibull. This will generally be true if the suspensions are randomly dispersed with the data. Note, however, the effect on the characteristic life, η. It went from 80 minutes without suspensions to 100 minutes with suspensions. The analysis resulting in 100 minutes is the correct method.

2.5 WEIBULL CURVE INTERPRETATION

Weibull curves may reveal clues about the failure mechanism, since different slopes imply different failure mechanisms.

If the slope is less than 1.0, reliability increases as the unit ages. This is often referred to as an infant mortality failure mode. Quality control and assembly problems may produce infant mortality failures. For instance, some gas turbine failures having slopes less than 1.0 are:

a. Improper augmentor liner repair — quality
b. Improper installation of temperature probes — misassembly
c. Fuel pump leaks due to installation problems — misassembly
d. Overhaul-related failures of various components — quality/misassembly
e. Electronic control failures — quality.

[1] Kapur and Lamberson, (1977). Reliability in Engineering Design. John Wiley and Sons, Inc., pp. 300.

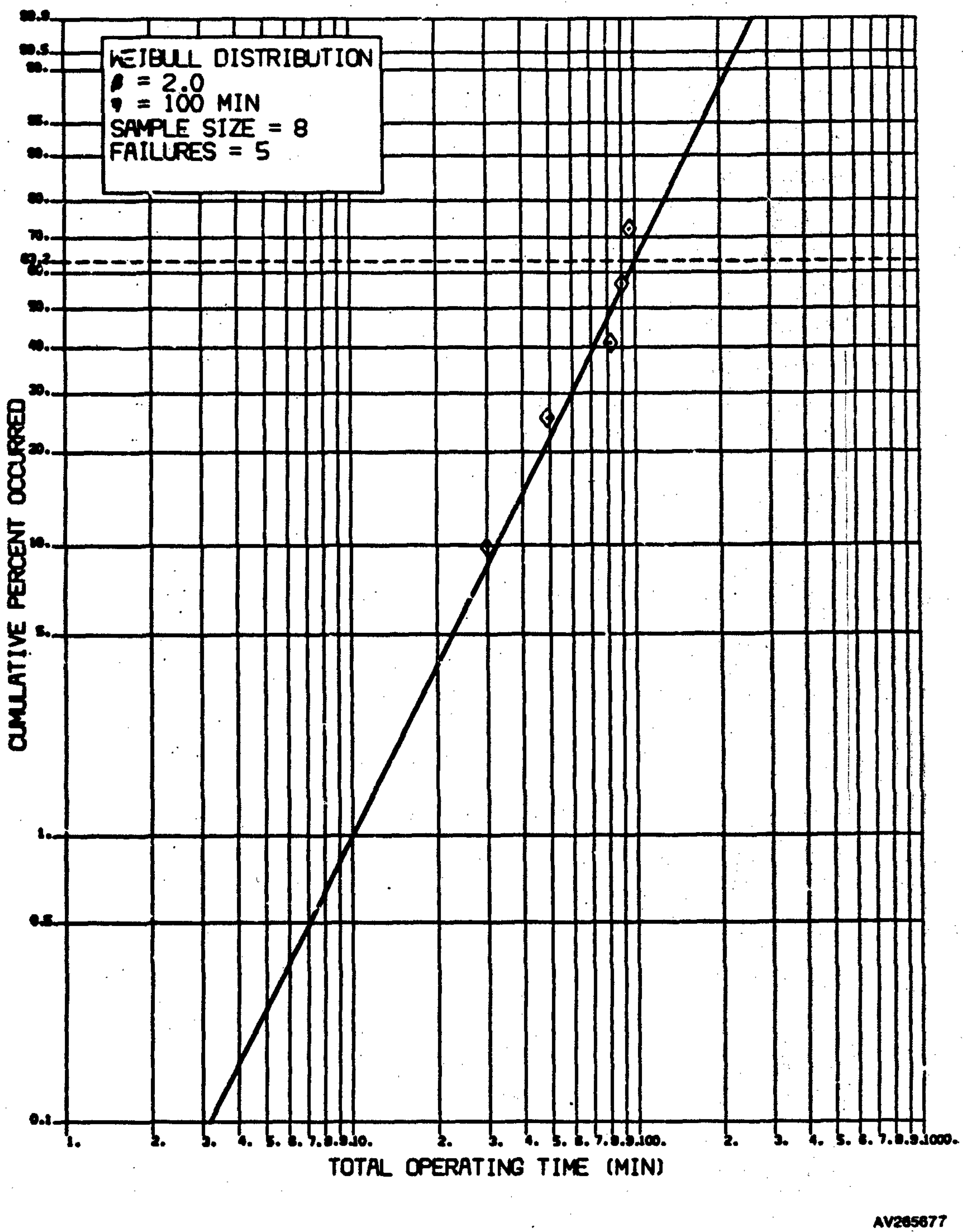


Figure 2.3. Rivet Failures With Suspensions

The exponential distribution is a special case of the Weibull distribution when $\beta = 1.0$. The exponential at times reflects original design deficiencies, insufficient redundancy, unexpected failures due to ingestion, or even product misuse. This would result in a constant failure rate condition. Some examples of Weibulls with slopes of 1 or near 1 are

a. Bearing cage failure
b. Temperature probe failure
c. Fuel control solenoid failure
d. Fuel oil cooler failure
e. Electronic engine control failure.

Slopes greater than 1 represent wearout modes. For shallow slopes like 1.8 to 3.0 there is more scatter in the failure data and therefore failure predictions will cover long timespans reflecting this uncertainty. As the slopes get steeper, failure times become more predictable. Some examples of Weibulls with slopes greater than 1 are:

a. Turbine vane wearout
b. Augmentor liner burnthrough
c. Temperature probe boss fatigue
d. Gearbox housing cracks
e. Augmentor flameholder cracks
f. Oil tube chafe through.

A slope, β, of 3.44 would approximate the familiar bell-shaped or normal curve, as indicated in Figure 2.4.

2.6 DATA INCONSISTENCIES AND MULTIMODE FAILURES

There are other subtleties in Weibull analysis which might signal problems. Examples are given that illustrate the following:

a. Failures are mostly low-time parts
b. Serial numbers of failed parts are close together
c. The data has a "dogleg" bend or cusp when plotted on Weibull paper
d The data has a gradual convex or concave bend on Weibull paper.

2.7 LOW-TIME FAILURES

Figure 2.5 is an example of low-time part failures on main oil pumps. Gas turbine engines are tested before being shipped to the customer, and since there were over 1000 of these engines in the field with no problems, what was going wrong? Upon examining the failed oil pumps it was found that they contained oversized parts. Something had changed in the manufacturing process which created this problem. The oversized parts caused an interference with the gears in the pump which resulted in failure. This was traced to a machining operation and corrected.

The point here is that low-time failures often indicate wearout (abnormal in this case) by having a slope greater than one when plotted. Low-time failures provide a clue to a production or assembly process change, especially when there are many successful high-time units in the field.

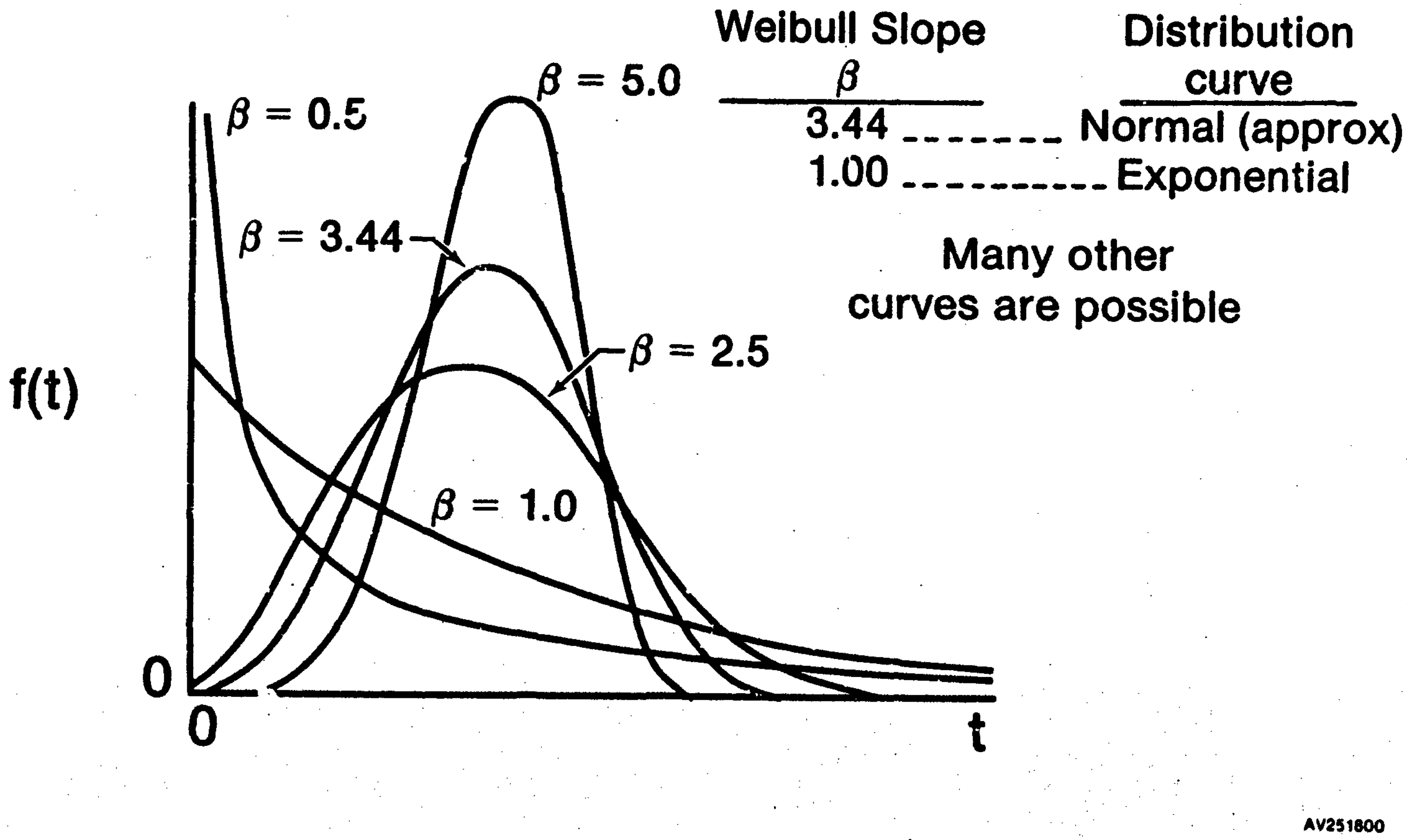


Figure 2.4. ***Failure Distribution Characteristic***

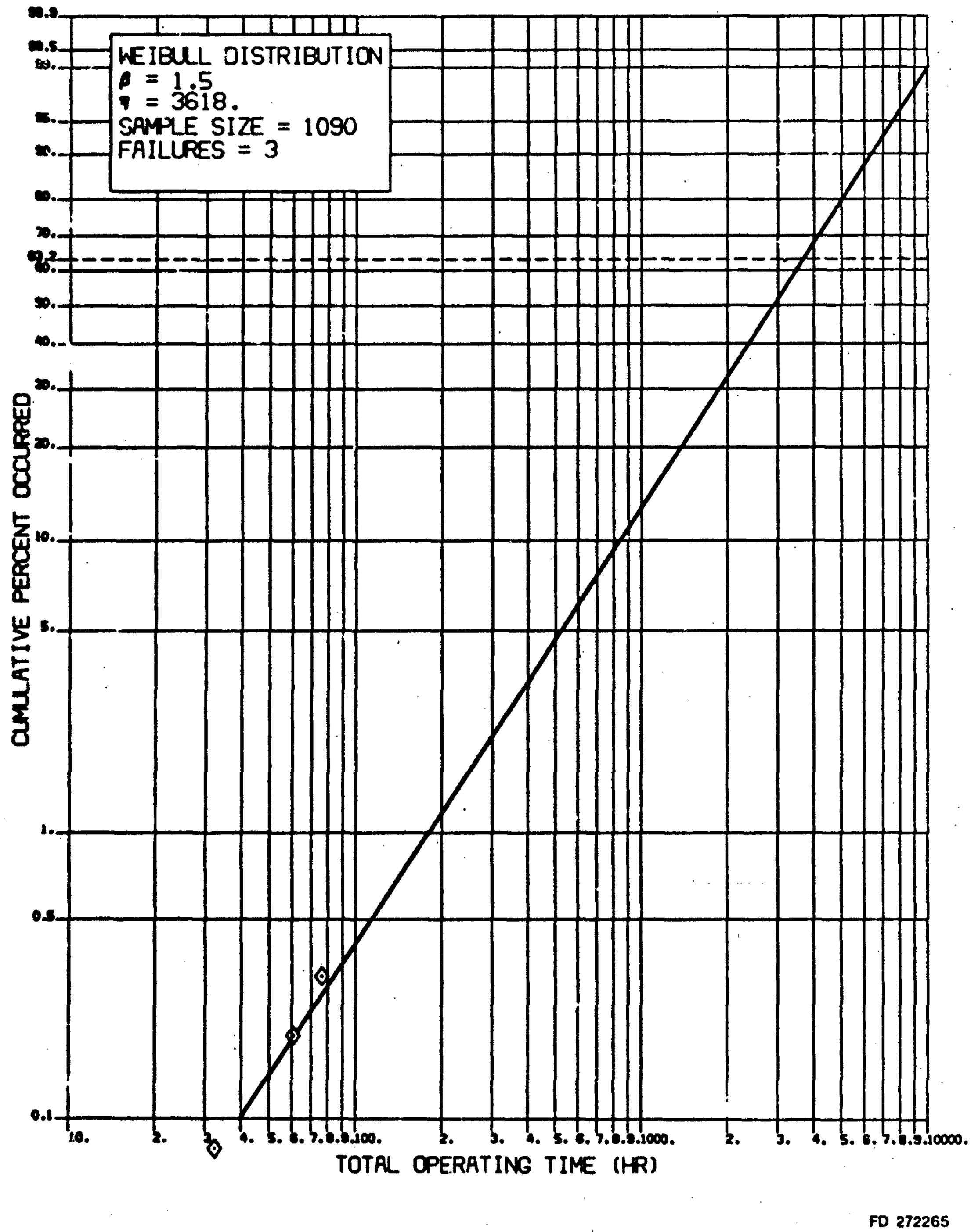


Figure 2.5. Main Oil Pumps

2.8 CLOSE SERIAL NUMBERS

The same reasoning can be extended to other peculiar failure groupings. For example, if failures occur in the middle of the time experience, that is, low-time units have no failures, mid-time units have failures, and high-time units have no failures, then a batch problem is suspected. Something may have changed in the manufacturing process for a short period of time and then changed back. The closeness of the serial numbers of the parts are a very definite clue to this type of problem.

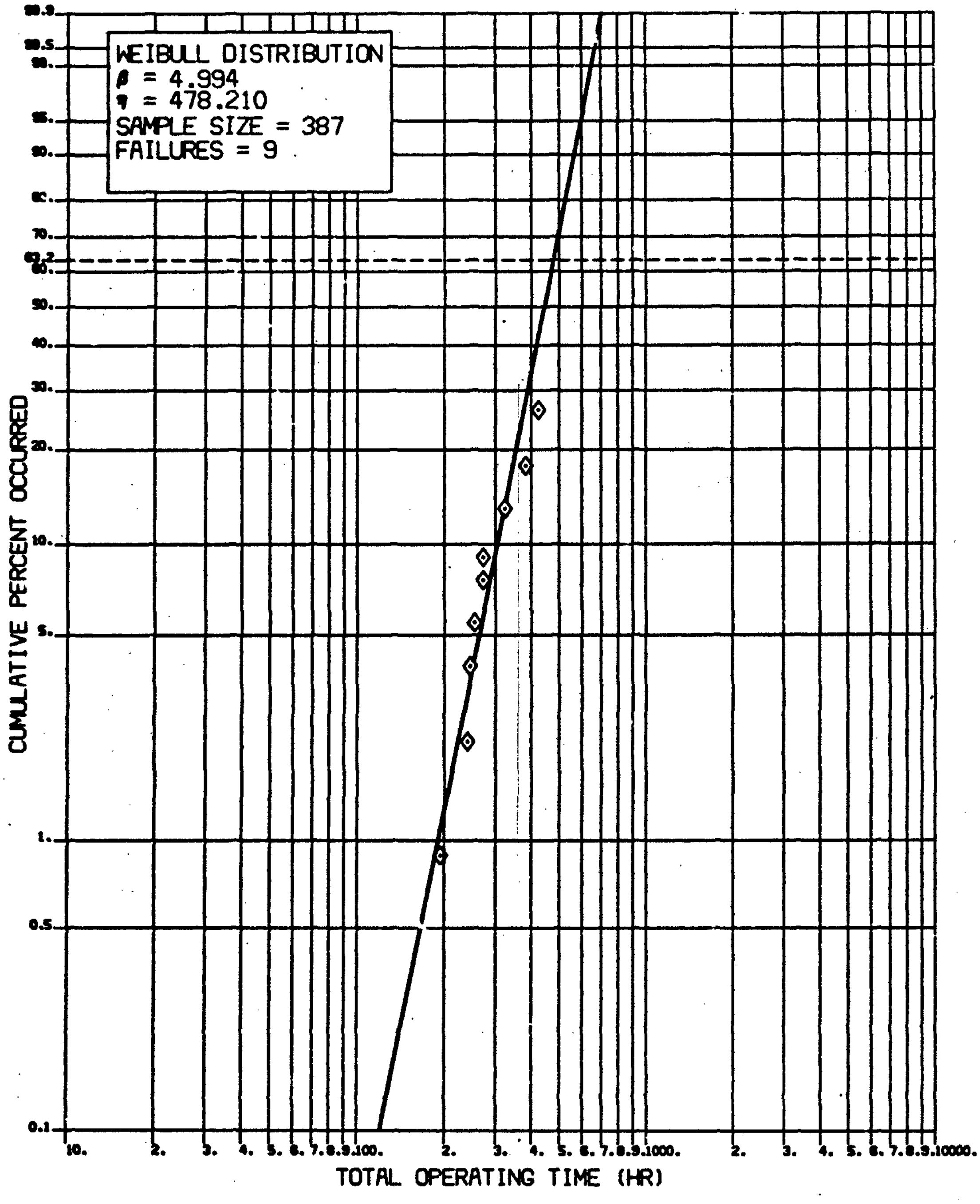
Figure 2.6 is a prime example of a process change which happened midstream in production. Bearings were failing in the augmentor pump. The failures had occurred in the 200 to 400 hour time frame. At least 650 units had more time than the highest time failure. These failures were traced to a process change that was incorporated as a cost reduction for manufacturing the bearing cages.

2.9 DOGLEG BEND

A Weibull plot containing a "dogleg bend" is a clue to the potential of multiple failure modes (see Figure 2.7). This was the case for a compressor start bleed system binding problem. Upon examination of the data, 10 out of 19 failures had occurred at one base. It was concluded that this base's location was contributing to the problem. The base was located on the ocean and the salt air was the contributing factor.

The data were broken apart and the two resultant Weibull charts are presented in Figures 2.8 and 2.9. Note that the fleet Weibull presented in Figure 2.8 is less than one, $\beta = 0.837$. This could be considered an infant mortality problem, while the ocean base Weibull, Figure 2.9, is more of a wearout failure mechanism with $\beta = 5.223$. This problem was related to lack of maintenance. More attention was given to this area and the problem was resolved.

The failures do not have to be associated with an environmental factor to cause a dogleg Weibull. In fact, they are usually associated with more than one failure mode. For instance, fuel pump failures could be due to bearings, housing cracks, leaks, etc. If these different failure modes are plotted on one Weibull plot, several dogleg bends will result. In cases where this occurs without prior knowledge, a close examination of the failures will have to take place for potential separation into different failure modes.



FD 272266

Figure 2.6. Weibull Plot for Augmentor Pump

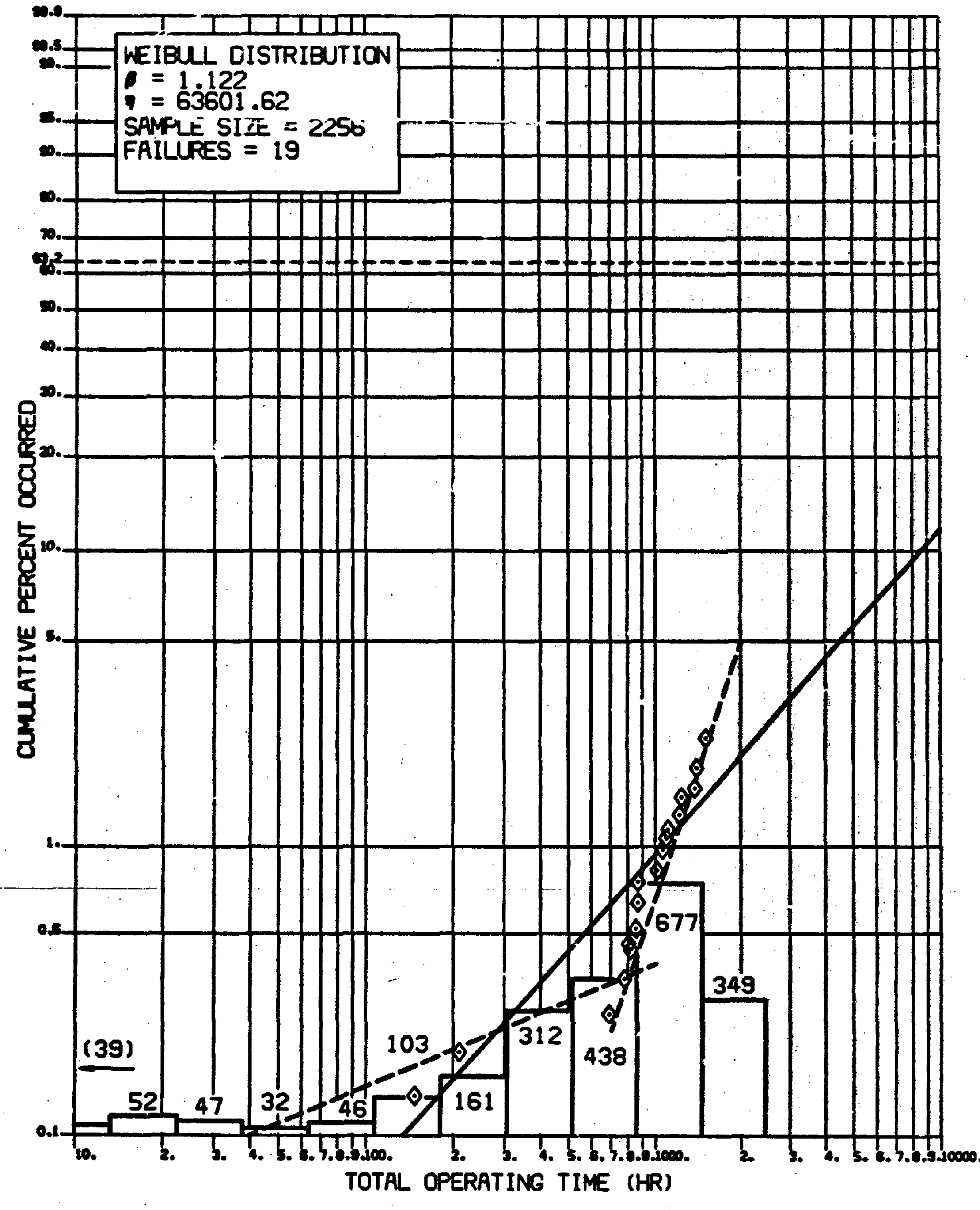
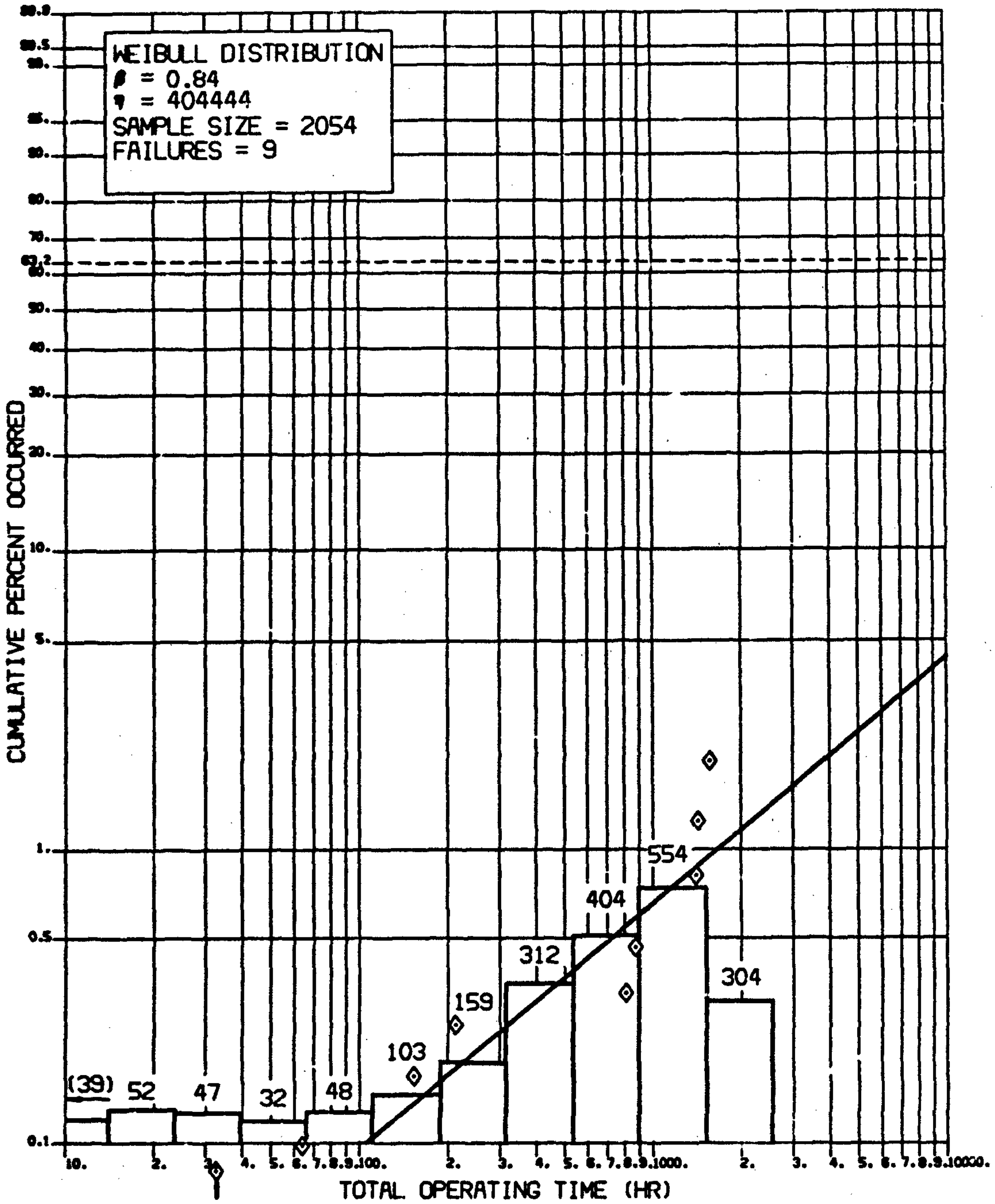


Figure 2.7. Compressor Start Bleed System



FD 272268

Figure 2.8. Compressor Start Bleed System, Excluding Ocean Base

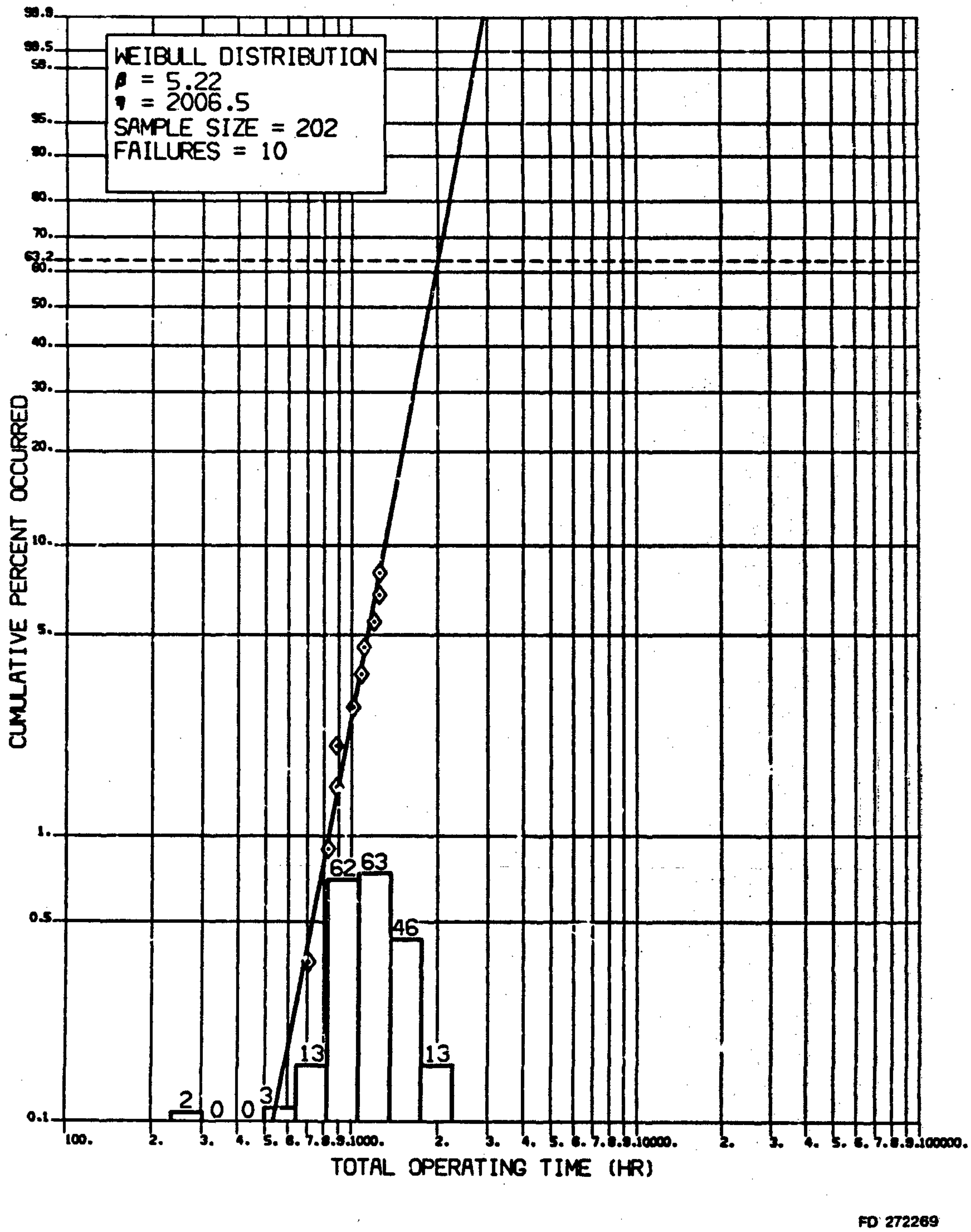


Figure 2.9. Compressor Start Bleed System, Ocean Base

2.10 CURVED WEIBULLS

In Section 2.1, the cumulative distribution function F(t) is presented for the Weibull. It was illustrated as:

$$F(t) = 1 - e^{((t - t_o)/\theta)^\beta}$$

where:

t = failure time
t_o = starting point or origin of the distribution.

In discussions so far, t_0 was assumed to be zero. When data are plotted on Weibull paper it quickly becomes obvious if the origin of time is not zero. The data will appear curved as illustrated in Figure 2.10 if the zero time origin is not true.

There are other reasons for poor fit (i.e., the data do not form a straight line on Weibull paper). For example, another distribution like a Normal, log Normal, etc. may better describe the data. If this is true, the distribution which best describes the data should be used.

But the data displayed in Figure 2.10 was from engine controls and there was no reason to suspect that the Weibull distribution could not be used to analyze it. There are a couple of ways to determine what adjustment is needed to make the data appear straight. First, there is an analytical method that can be used to establish t_o. The equation is:

$$t_o = t_2 - \frac{(t_3 - t_2)(t_2 - t_1)}{(t_3 - t_2) - (t_2 - t_1)}$$

Where t_1 is the first failure time, t_2 is the time corresponding to the linear halfway distance on the vertical axis between the first and last failure, and t_3 is the last failure time. This is illustrated in Figure 2.11. The values for t_1, t_2, and t_3 are:

t_1 = 16.9 hours ~ first failure

t_2 = 42.0 hours ~ halfway failure

t_3 = 389.0 hours ~ last failure

$$t_o = 42.0 - \frac{(389.0 - 42.0)(42.0 - 16.9)}{(389.0 - 42.0)-(42.0 - 16.9)}$$

t_o = 42.0 - 27.1

t_o = 14.9 hours.

If t_o is positive, it implies that the origin starts after zero; if negative, before zero. In other words, there is a time, in this case approximately 15.0 hours, in which the control would not be expected to fail.

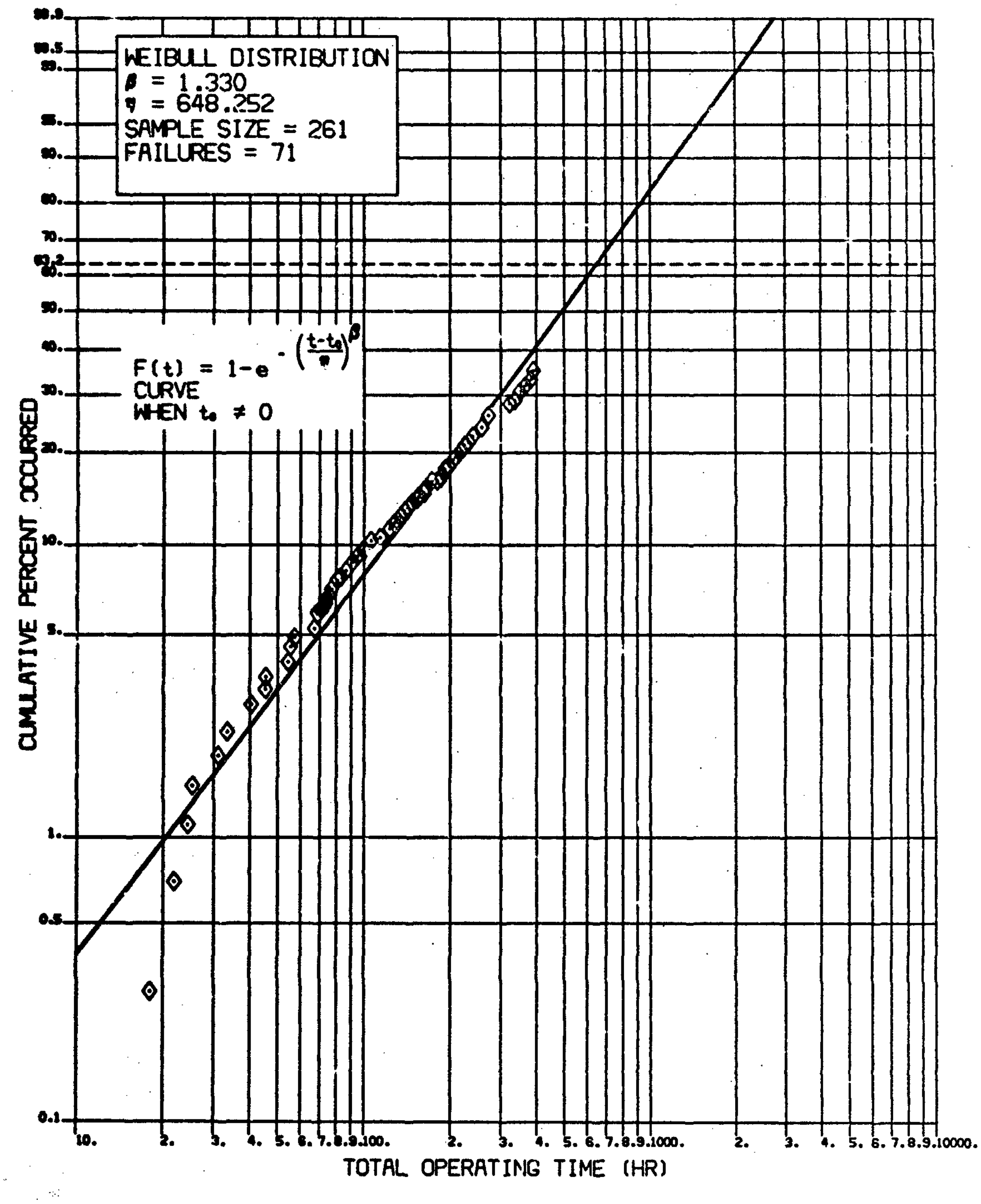


Figure 2.10. A Curved Weibull Needs t_o Correction

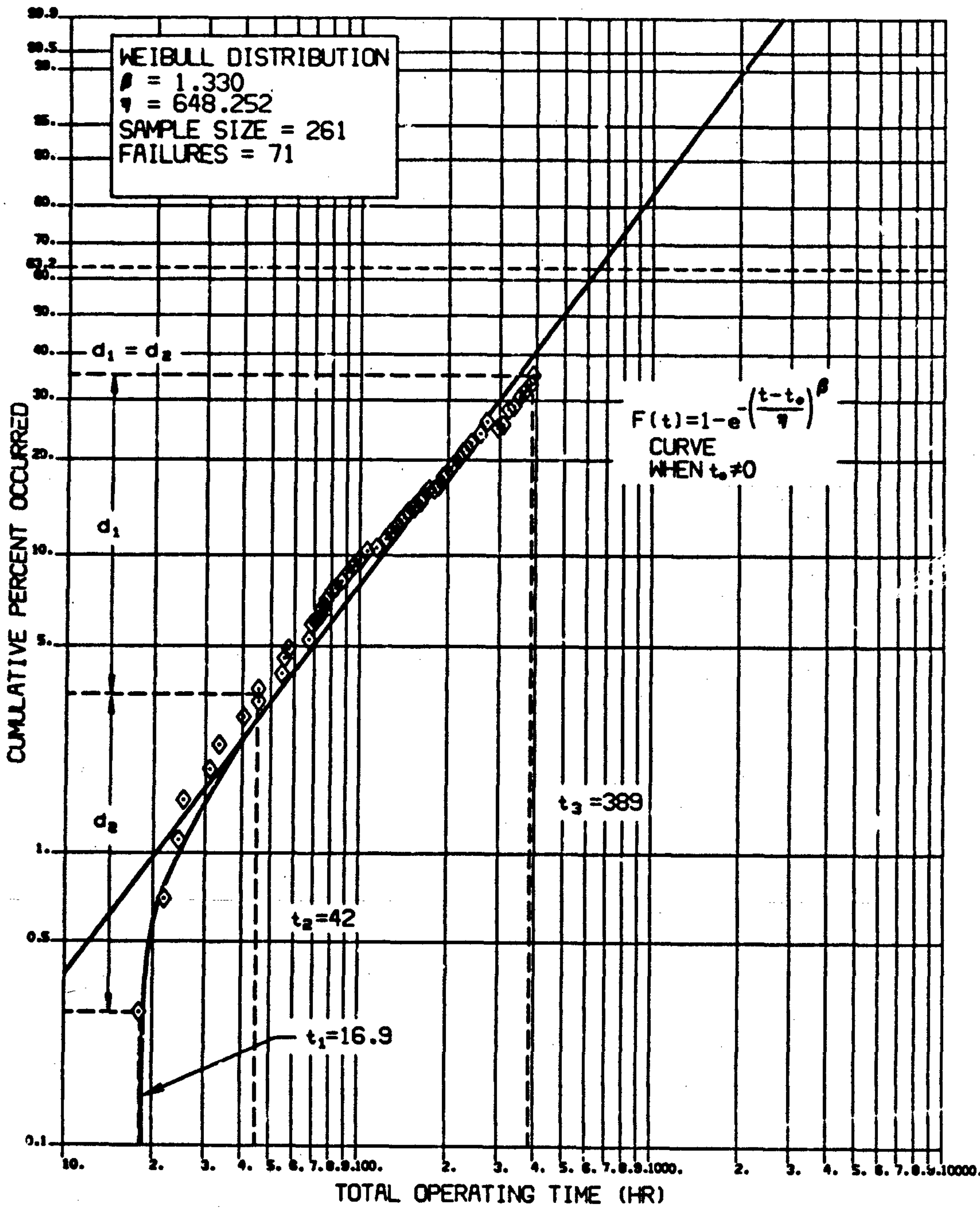


Figure 2.11. Plotting t_o Correction

Upon questioning the vendor, this was not found to be true. The vendor was actually testing the units for about 15 hours prior to shipment and discarding or repairing failed units. This made the distribution appear to be truncated at 15 hours with a zero probability of failure before that time. Subtracting 15 hours from each of the failure times will adjust the curve for the absence of this time. The resultant curve is plotted in Figure 2.12.

The corrected curve provides a more accurate prediction of the probability of failure. However, to determine distribution percentiles like the B.1 life or B1 life, one has to add the 15 hours to the time read from the Weibull plot.

For example, to determine the time to failure for the 1/100 unit (often referred to as the B1 life), one would read the 1 percentage point of 8 hours from Figure 2.12 and then add 15 hours to it. That is, the B1 life is estimated to be 23 hours.

The second way to correct a curved Weibull uses a simplistic approach. When the curved Weibull becomes fairly perpendicular to the horizontal scale, extend the curved Weibull vertically through the time scale. Where it intersects, simply read the curve and subtract or add the time. Trying this technique on Figure 2.11 confirms that 15 hours is a good estimate. By eye, this curve would be considered convex; therefore, a subtraction of time would be required. Data plotted on Weibull paper that curves in the other direction (concave) would require adding time to each point. The amount of time to be added would be found with either of the above procedures.

2.11 PROBLEMS

Problem 2-1:

Fatigue specimens were put on test. They were all tested to failure and the failure times were 150, 85, 250, 240, 135, 200, 240, 150, 200, and 190 hours.

a. Construct a Weibull and determine its slope, β, and characteristic life, η.
b. Would you have expected the derived slope for fatigue specimens?
c. If you were quoting the $B_{1.0}$ life, what would the value be?

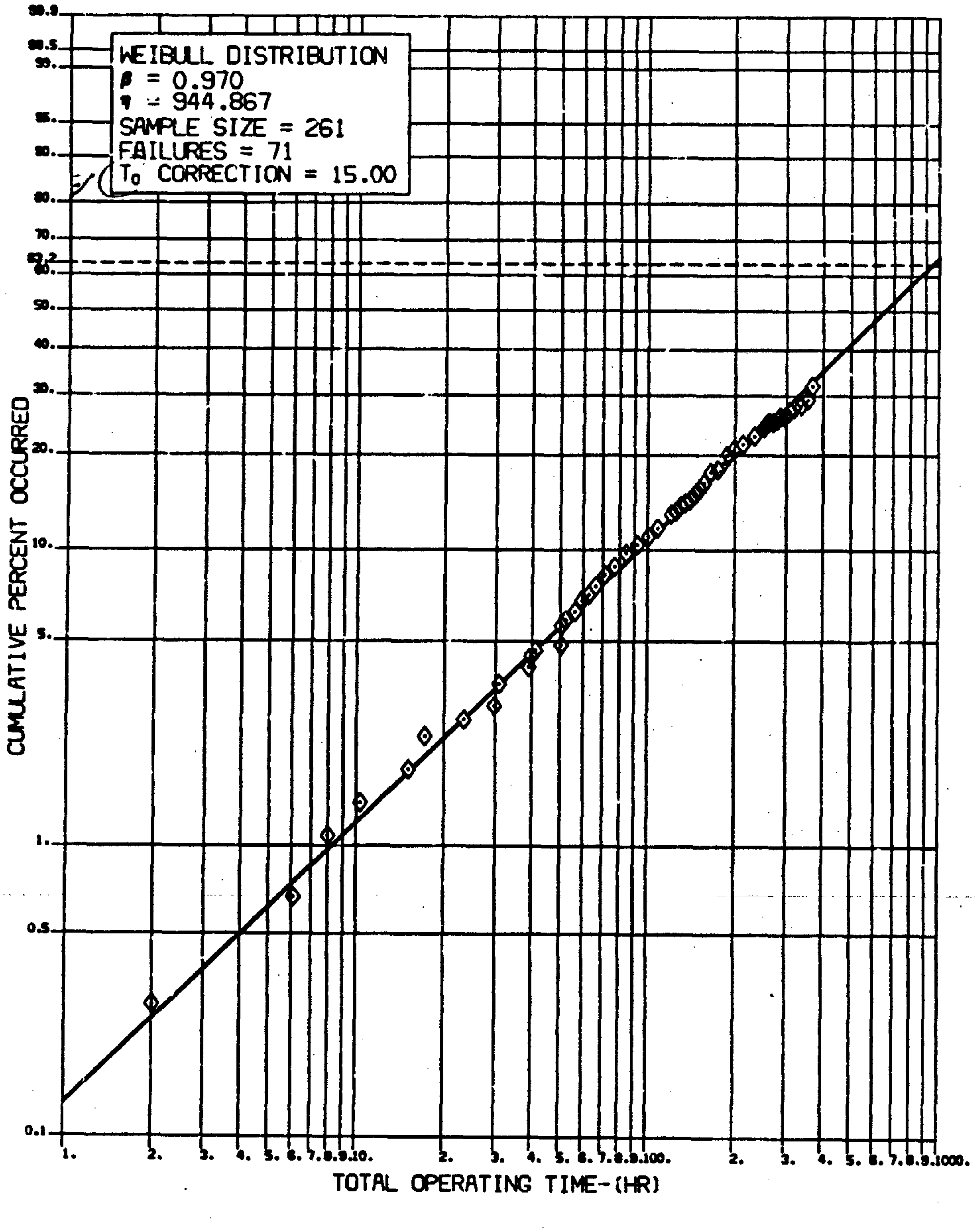


Figure 2.12. t_0 Correction Applied

Problem 2-2:

There were five failures of a part in service. The information on these parts is

Serial Number	*Time (hours)*	*Comment*
831	9.0	Failure
832	6.0	Failure
833	14.6	Suspension
834	1.1	Failure
835	20.0	Failure
836	7.0	Suspension
837	65.0	Failure
838	8.0	Suspension

a. Construct a Weibull with suspensions included and determine its slope, β, and characteristic life, η.

b. What is the failure mode?

c. Are there other clues which may lead to an answer to the problem?

Problem 2-3:

The following set of failure points will result in a curved Weibull: 90, 130, 165, 220, 275, 370, 525, and 1200 hours.

a. What value is needed to straighten the Weibull?
b. Will the value found in "a" be added or subtracted from the failure values?

Solutions to these problems are in Appendix J.

CHAPTER 3

WEIBULL RISK AND FORECAST ANALYSIS

3.1 FOREWORD

One of the major uses of Weibull analysis is to predict the number of occurrences of a failure mode as a function of time. This projection is important because it gives management a clear view of the potential magnitude of a problem. In addition, if this prediction is made for different failure modes, management is able to set the priority for the solution of each problem.

In this chapter the use of the Weibull probability distribution function in predicting the occurrences of a failure mode is explained. The additional input needed for risk analysis will be covered, and several examples are presented to explain further the techniques involved.

It should be emphasized that the forecast analysis is only as good as the failure data. The data should be examined closely to ensure that they are from a single failure mode and will fit a Weibull distribution.

3.2 RISK ANALYSIS DEFINITION

A risk analysis calculates the number of incidents projected to occur over some future period.

3.3 FORECASTING TECHNIQUES

The observed failures and the population of units that have not failed are used to obtain the Weibull failure distribution, as discussed in Chapter 2. The following additional input is needed for forecasting:

a) Usage rate per unit per month (or year, day, etc.)

b) Introduction rate of new units (if they are subject to this same failure mode)

With this information a risk analysis can be produced. The techniques used to produce the risk analysis can vary from simple calculations to those involving Monte Carlo simulation. Monte Carlo simulation is only required when complications arise in the risk analysis. These will be explained in the following sections.

3.4 CALCULATING RISK

Risk calculations are described in three sections:

- Present risk
- Future risk when failed units are not fixed
- Future risk when failed units are fixed.

3.5 PRESENT RISK

The simplest case arises when there are no new units (no production) and no replacement of failed units. If there is a population of N items and each has accumulated t hours or cycles, the expected number of failures from this population is the probability of failure by time t multiplied by the number of units, N. Therefore, for a Weibull distribution this becomes

$$\text{Expected number of failures} = F(t) \cdot N = \left(1 - e^{-(t/\eta)^{\beta}}\right) N. \tag{3.1}$$

Equation 3.1 can be used immediately to calculate the following:

There are 25 units in a population: 5 units have accumulated 1000 hours of operational time, 5 units have accumulated 2000 hours, 5 units have accumulated 3000 hours, 5 units have accumulated 4000 hours, and 5 units have accumulated 5000 hours. Assume that the population is subject to a Weibull failure mode with $\beta = 3.0$ and $\eta = 10000$ hours. The question is, 'What is the cumulative expected number of failures from time 0 to now for this population?" Figure 3.1 is the Weibull failure distribution with the cumulative probability of failure by each time on the units as illustrated. Table 3.1 summarizes the calculations involved.

TABLE 3.1. PRESENT RISK

Number(N) of Units	*Time (t) on Each Unit*	*F(t)*	*F(t) · N*	*Example of Calculation:*	
				F(t)	$= 1 - e^{-(t/\eta)^{\beta}}$
5	1000	0.001	0.005	F(1000)	$= 1 - e^{-(1000/10000)^3}$
5	2000	0.008	0.040		$= 1 - e^{-(0.1)^3}$
5	3000	0.027	0.135		$= 1 - e^{-0.001}$
5	4000	0.062	0.310		$= 1 - 0.999$
5	5000	0.117	0.585	F(1000)	= 0.001
			Sum = 1.075		

The value of F(t) can also be read directly from the Weibull Cumulative Probability Plot. (See Figure 3.1.)

The cumulative expected number of failures in this case is 1.075.

3.6 FUTURE RISK WHEN FAILED UNITS ARE NOT FIXED

Given the same 25 units as in Table 3.1, the expected number of failures over the next 12 months can be calculated. Assume that one of the 4000 hour units has just failed. Since it is assumed that failed units will not be replaced, it will be omitted from the population for the calculation of future risk.

Yearly usage of each unit will be 300 hours. The future risk will be composed of the risk of the 1000-hour units failing by 1300 hours, plus the risk of the 2000 hour units failing by 2300 hours, plus the risk of the 3000 hour units failing by 3300 hours, etc.

In general, if a unit has accumulated t hours to date without failure, and will accumulate u additional hours in a future period, then that unit's contribution to the total future risk is:

$$\frac{F(t+u) - F(t)}{1 - F(t)} \tag{3.2}$$

where $F(t) = 1 - e^{-(t/\eta)^\beta}$ is the probability of the unit failing in the first t hours of service, assuming it follows a Weibull failure distribution. If F(t) is much less than 1.0, equation (3.2) is approximately equal to

$$F(t + u) - F(t) \tag{3.3}$$

Table 3.2 summarizes the future risk calculations for the population of 25 units, with one failed unit at 4000 hours.

Hence the expected number of failures from this population over the next 12 months is:

$$\text{Failures} = 5(0.0012) + 5(0.0041) + 5(0.0089) + 4(0.0154) + 5(0.0236) = 0.2506$$

3.7 FUTURE RISK WHEN FAILED UNITS ARE REPAIRED

The calculation of the number of failures that will occur over some future time interval when the failed units will be repaired and returned to service involves the same concepts as when units are not fixed. When the probability of failure of a unit over the time interval in question is small (on the order of 0.5 or less), the techniques of Paragraph 3.6 can be applied. In cases where the probability of failure is greater than about 0.5, the chance of more than one failure over the same time interval becomes significant. Then, the expected number of failures may be calculated using published tables[1], complex mathematical formulas, or Monte Carlo simulation methods.

3.8 THE USE OF SIMULATION IN RISK ANALYSIS

The calculation of risk is easy for the simple case of a population with no inspections, no production added, and no retrofits. Of course, even simple risk analysis can become complicated by the volume of calculations involved. In this case, a computer program automating the calculations is useful.

In some instances, a part's service life will depend on decisions to be made in the future which will be dependent on a Weibull distribution. Since only the probability of this outcome may be known, a powerful tool known as Monte Carlo simulation is useful. Monte Carlo simulation enables an analyst to build a computer model of the decision plan as it affects a part's service life. It may include scheduled part inspections, random events such as the extensive wear of a particular part and its replacement, as well as the addition of new units into the field.

The effect of scheduled inspection on risk is straightforward. If a part is inspected and removed from service, it no longer contributes to the fleet's risk. If it continues in service, it continues to contribute to the fleet's future risk.

As an example, the methodology used in a Monte Carlo simulation is described for the case of three failure modes and a scheduled inspection. In this case, the number of failures occurring in each mode before the scheduled inspection is desired. However, the occurrence of any one failure mode will not affect any other mode.

[1] WHITE, J. S. (1964), "Weibull Renewal Analysis," in *Proceedings of the Aerospace Reliability and Maintainability Conference, Washington, D. C., 29 June — 1 July 1964*, New York: Society of Automotive Engineers, 639-657.

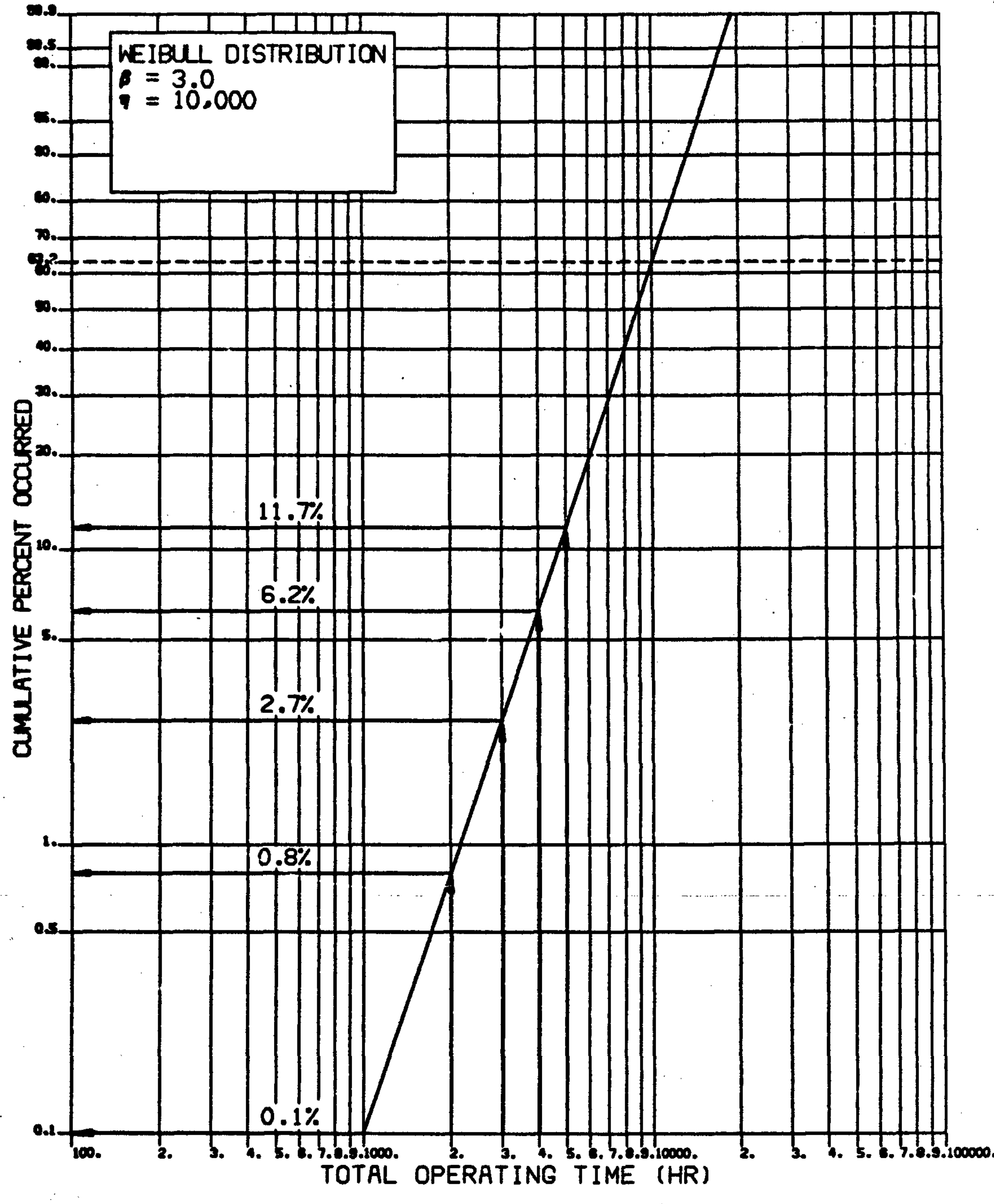


Figure 3.1. Risk Analysis With Weibulls

TABLE 3.2. FUTURE RISK

Number of Units (N)	Current Time on Each Unit (t) (hr)	Time on Each Unit at Year's End (t+u) (hr)	F(t+u)	F(t)	Each Unit's Risk $\frac{F(t+u)-F(t)}{1-F(t)}$
5	1000	1300	0.0022	0.0010	0.0012
5	2000	2300	0.0121	0.0080	0.0041
5	3000	3300	0.0353	0.0266	0.0089
4	4000	4300	0.0764	0.0620	0.0154
5	5000	5300	0.1383	0.1175	0.0236

The following procedure is performed for each unit in the population. Using random numbers that are evenly (uniformly) distributed between 0 and 1 and the three Weibull failure distributions, generate a time-to-failure for each failure mode. See Figure 3.2. The following equation is used to calculate the time to failure:

$$\text{time to failure} = \eta \left[\ln\left(\frac{1}{1 - \text{random number}}\right)\right]^{1/\beta} \quad (3.4)$$

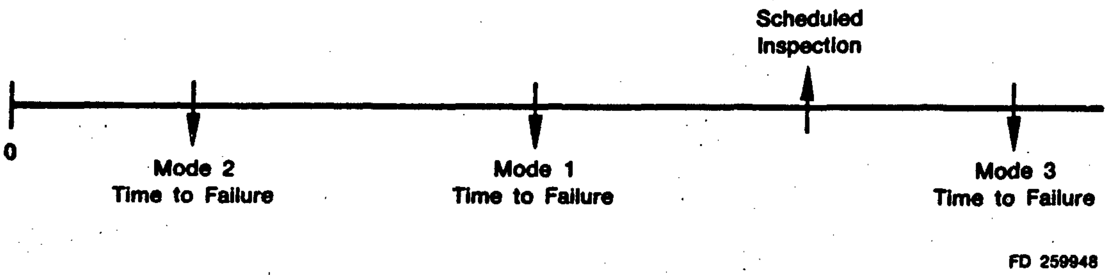


Figure 3.2. Simulation Logic — First Pass

Advance the simulator to the first event; if this event is a failure, note the cause, and regenerate a new time to failure for this mode. See Figure 3.3.

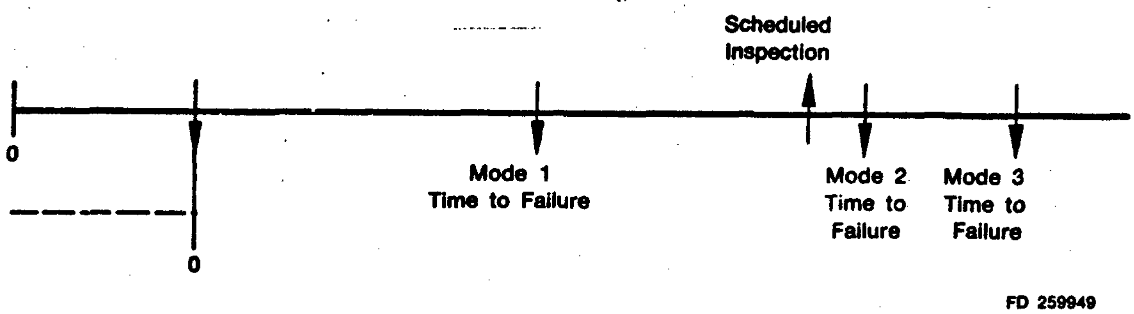


Figure 3.3. Simulation Logic — Second Pass

Continue this process until the scheduled inspection is reached. The number of failures of each mode is recorded and the simulation is repeated. After many repetitions of this process, each using a different set of random numbers, the results are averaged to give the expected risk.

A more detailed example utilizing these principles is given in Section 3.12.

3.9 CASE STUDIES

Several case studies in the use of the ideas developed in the previous sections are now presented. The first two examples, Sections 3.10 and 3.11, illustrate the direct calculation of risk without simulation. The case study in Section 3.12 uses Monte Carlo simulation.

3.10 CASE STUDY 1: BEARING CAGE FRACTURE

Bearing cage fracture times of 230, 334, 423, 990, 1009, and 1510 hours were observed. The population of bearings within which the failures occurred is shown in Figure 3.4. A Weibull analysis similar to those described in Chapter 2 was followed to obtain the Weibull failure distribution for bearing cage fracture (Figure 3.5). From this distribution plot we can see that the B_{10} life (time at which 10% of the population will have failed) is approximately 2430 hours. This was much less than the B_{10} design life of 8000 hours, so a redesign was undertaken immediately. Additionally, management wanted to know how many failures would be observed before this redesign entered the field.

The risk questions and solutions are:

1. How many failures could be expected by the time units had reached 1000 hours?

   Calculate the number of units that will fail by 1000 hours, assuming failed units are not replaced. Enter the x-axis of the Weibull plot (Figure 3.5) and read at 1000 hours that approximately 1.3% of the population is expected to fail. That is, after the entire population of 1703 bearings reach 1000 hours each, 1703 (0.013) = 22 bearings would be expected to have failed.

2. How many failures could be expected in the next year?

   Utilizing the methodology explained in Section 3.6 and applying Equation 3.2 with a monthly utilization of 25 hours or 12(25) = 300 hours in one year results in the calculations shown in Table 3.3. Thus about 12 more failures can be expected in the next 12 months.

3. How many failures could be expected when 4000 hours had been accumulated on each bearing if we instituted a 1000 hour inspection? A 2000 hour inspection? No inspection?

   From the answer to Question 1, the probability of a bearing failure by 1000 hours is 0.013. Therefore, if it is assumed that each 1000 hour inspection makes the bearing "good as new" relative to cage fracture, there is a total expectation of failure for each bearing by 4000 hours of approximately 0.013 + 0.013 + 0.013 + 0.013 = 0.052. So, if all 1703 bearings ran to 4000 hours with 1000 hour inspections, 0.052(1703)= 89 failures can be expected.

On the other hand, if there is a 2000 hour inspection, the probability of failure by 2000 hours is 0.065. Using the same approach as in the previous paragraph, by 4000 hours about 0.065 + .065 = 0.13 failures would be expected for each bearing. Therefore, the expected number of failures with a 2000 hour inspection would be 0.13(1703)=221.

Now suppose no inspections were made until 4000 hours, at which time the bearing will be retired. Again utilizing the Weibull in Figure 3.5, the probability of failure by 4000 hours is 0.28. Therefore, by the time all 1703 of the bearings have been retired, 0.28(1703)=477 will have failed.

3.11 CASE STUDY 2: BLEED SYSTEM FAILURES

Nineteen bleed system failures have been noted and the times and geographical locations of these failures are listed in Table 3.4. The high incidence at air base D prompted a risk analysis to determine the cumulative number of incidents to be expected over the next year at air base D.

A Weibull analysis of the fleet failures *excluding* air base D (Figure 3.6), indicates a decreasing failure rate phenomenon, that is, $\beta < 1.0$. But a Weibull analysis of the failures at air base D (Figure 3.7) indicates a rapid wearout characteristic. From comparison of the plots it seems that the bases are significantly different. It is shown in Chapter 7 that the two failure distributions may be proven statistically to be significantly different.

Since the probability of failure, excluding air base D, was quite low by 4000 hours (the life limit of the part) for the fleet, a risk analysis for air base D only was requested.

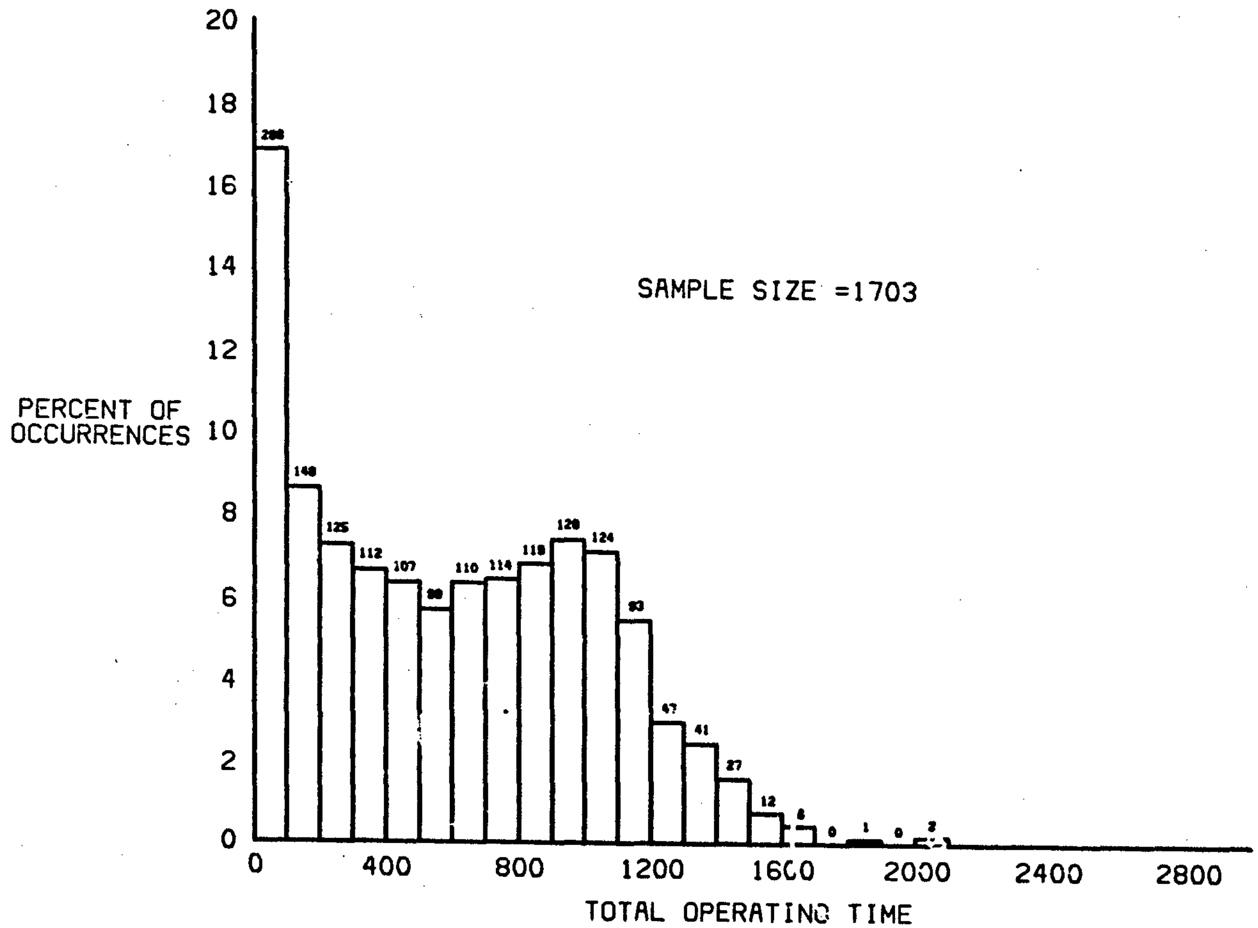
The risk questions are:

1) What is the expected number of incidents in the next year and a half with a usage of 25 hours per month?

   Using the histogram of the times on each bleed system at air base D (Figure 3.8), set up the calculation as before (Table 3.5). Over the next 18 months, 56 failures can be expected using a 25 hours per month utilization rate.

2) If the usage drops to 20 hours per month immediately, how many fewer failures can be expected?

   Changing the utilization rate to 20 hours per month will change the calculation of expected risk. The new risk over the next 18 months is given in Table 3.6. About 42 failures, or about 13 fewer than for a utilization rate of 25 hours per month, are predicted.



FD 271856

Figure 3.4. Bearing Population

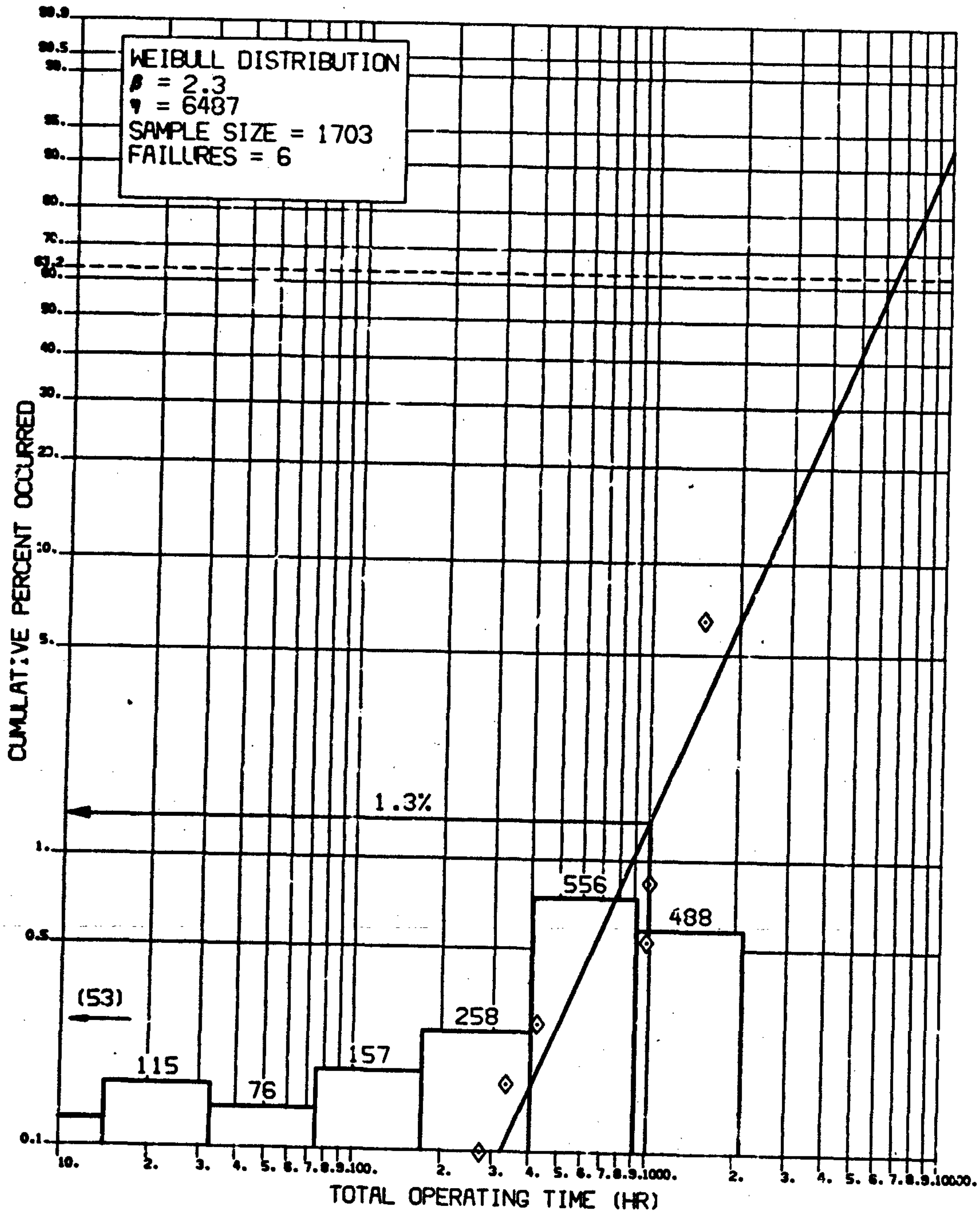


Figure 3.5. Bearing Cage Fracture

TABLE 3.3. BEARING RISK AFTER 12 MONTHS

Number of Units (N)	*Current Time on Each Unit (t)*	*Time on Each Unit at Year's End (t+u)*	*F(t)*	*F(t+u)*	*Each Unit's Risk* $\frac{F(t+u)-F(t)}{1-F(t)}$	*Total Risk* $\frac{F(t+u)-F(t)}{1-F(t)} \cdot N$
283	50	350	0.0000	0.0012	0.0012	0.3480
148	150	450	0.0002	0.0022	0.0020	0.2963
125	250	550	0.0006	0.0034	0.0029	0.3607
112	350	650	0.0012	0.0051	0.0038	0.4301
107	450	750	0.0022	0.0070	0.0049	0.5193
99	550	850	0.0034	0.0093	0.0059	0.5859
110	650	950	0.0051	0.0121	0.0070	0.7731
114	750	1050	0.0070	0.0151	0.0082	0.9325
119	850	1150	0.0093	0.0186	0.0094	1.1148
128	950	1250	0.0121	0.0225	0.0106	1.3558
124	1050	1350	0.0151	0.0268	0.0118	1.4691
93	1150	1450	0.0186	0.0315	0.0131	1.2214
47	1250	1550	0.0225	0.0366	0.0144	0.6790
41	1350	1650	0.0268	0.0422	0.0158	0.6473
27	1450	1750	0.0315	0.0481	0.0172	0.4631
12	1550	1850	0.0366	0.0545	0.0185	0.2225
6	1650	1950	0.0422	0.0613	0.0200	0.1197
0	1750	2050	0.0481	0.0685	0.0214	0.000
1	1850	2150	0.0545	0.0761	0.0228	0.0228
0	1950	2250	0.0613	0.0841	0.0243	0.0000
2	2050	2350	0.0685	0.0925	0.0258	0.0516
						Sum = 11.613

TABLE 3.4. BLEED SYSTEM FAILURES BY AIR BASE

Air Base	*Hours at Failure*
A	153
B	872
C	1568
A	212
D	1198
D	884
A	1428
C	808
D	1251
D	1249
C	1405
D	708
D	1082
D	884
D	1105
D	828
D	1013
E	64
F	32

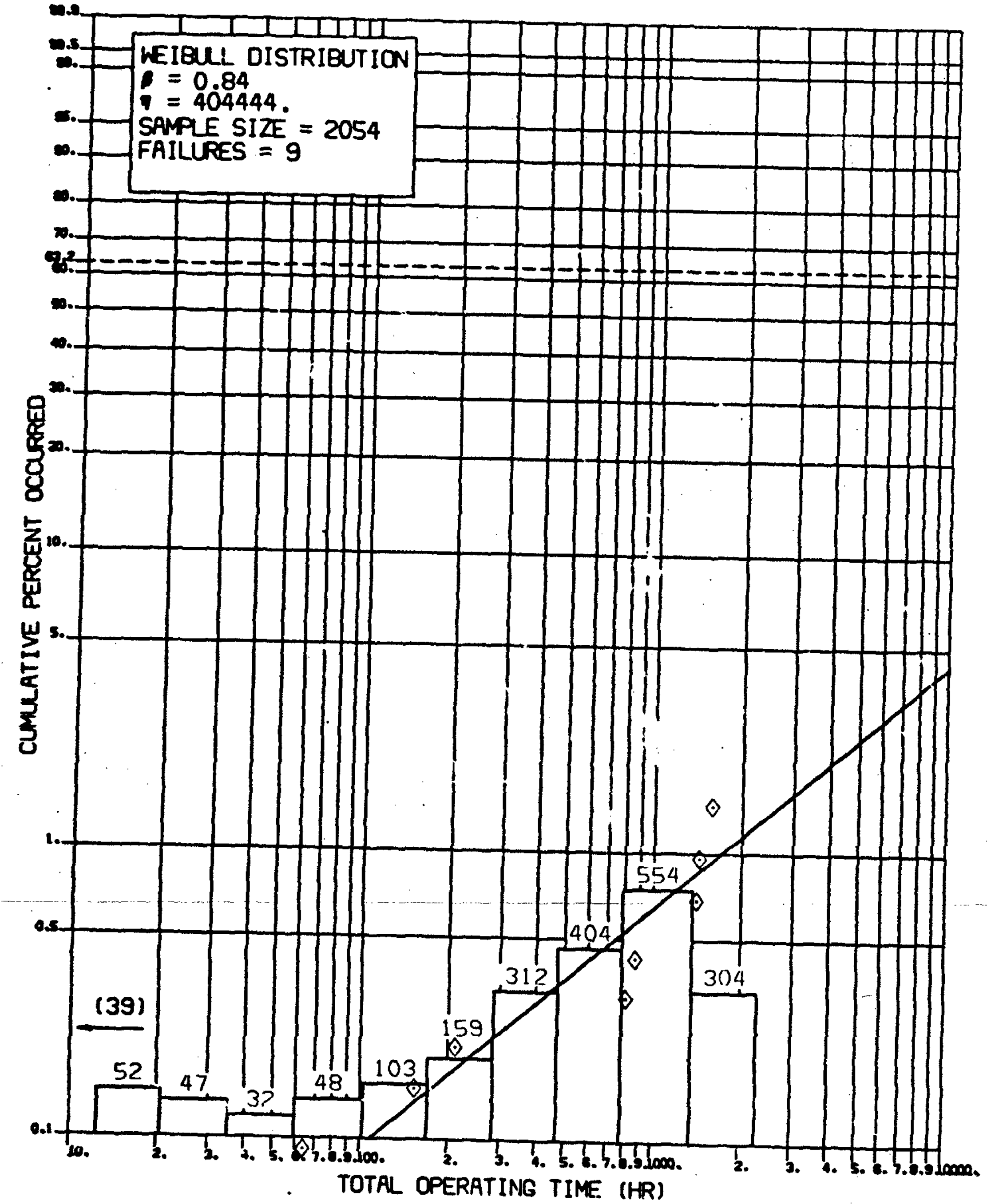
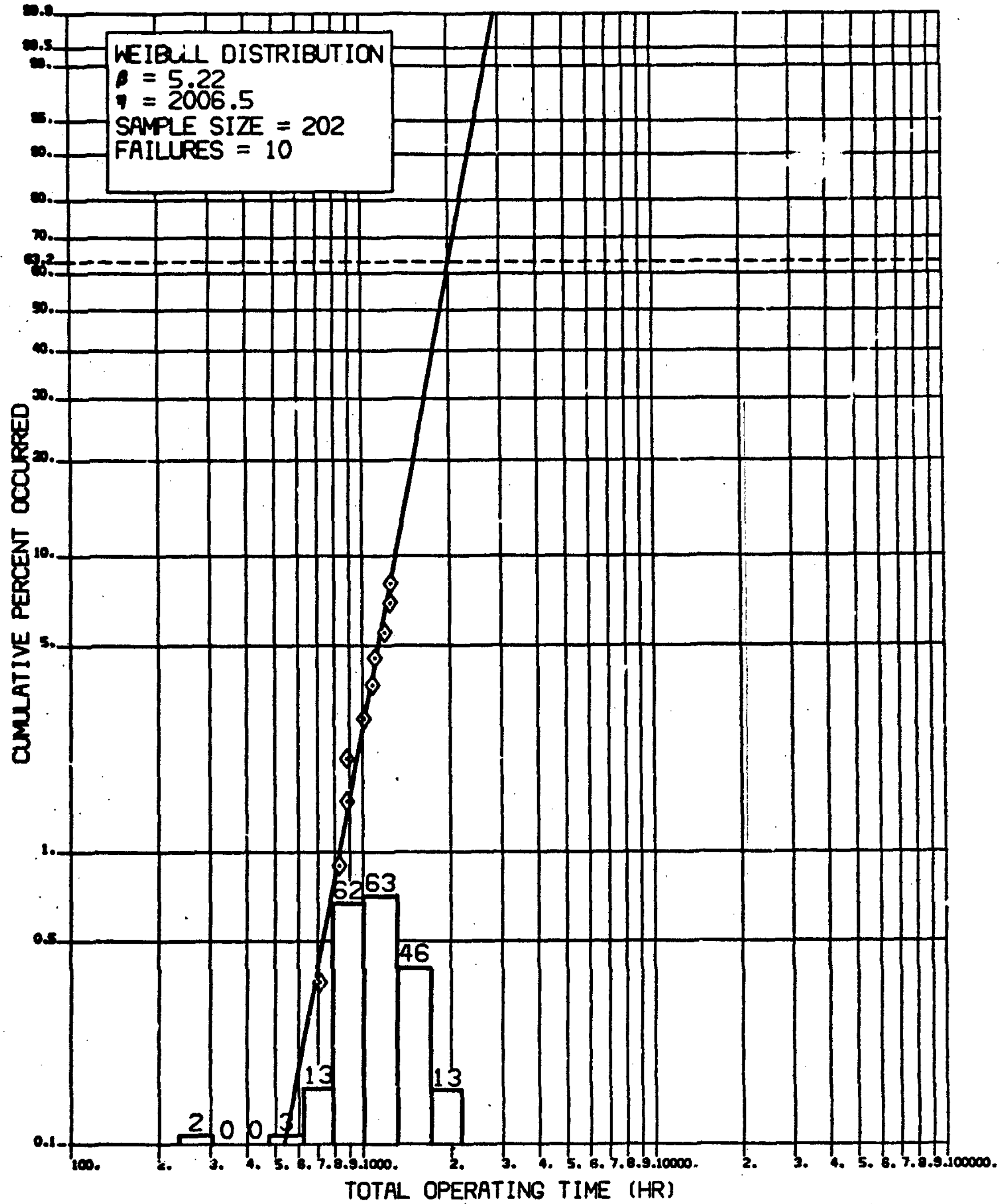


Figure 3.6. Bleed System Failure Distribution Excluding Air Base D



FD 271858

Figure 3.7. Bleed System Failure Distribution At Air Base D

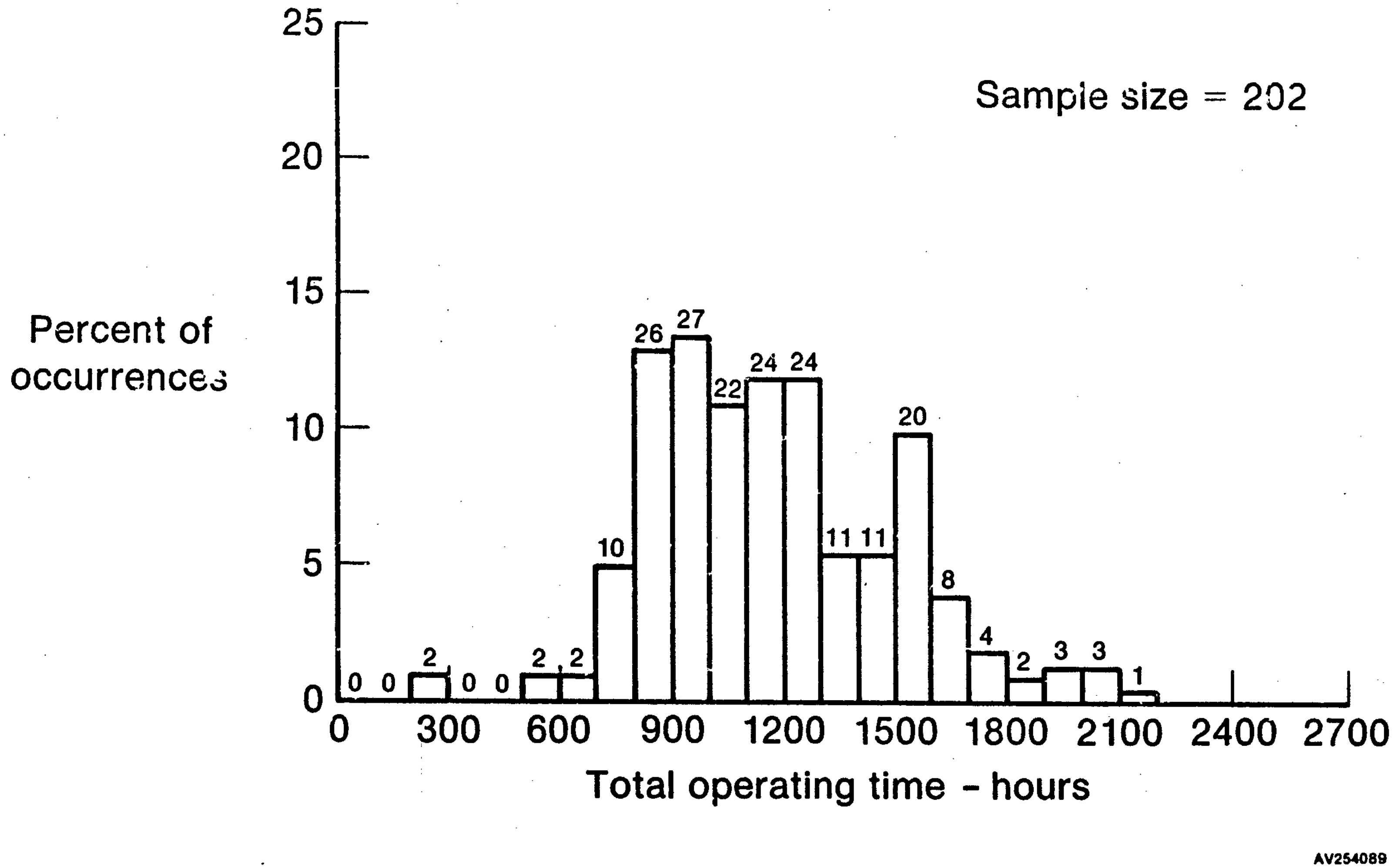


Figure 3.8. Bleed System Population - Air Base D

TABLE 3.5. BLEED SYSTEM RISK AFTER 18 MONTHS

(Utilization Rate 25 Hours Per Month)

Number of Units (N)	*Current Time on Each Unit (t)*	*Time on Each Unit in 18 Months (t+u)*	*F(t)*	*F(t+u)*	*Each Unit's Risk* $\frac{F(t+u)-F(t)}{1-F(t)}$	*Total Risk* $\frac{F(t+u)-F(t)}{1-F(t)} \cdot N$
0	50	500	0.0000	0.0007	0.0007	0.0000
0	150	600	0.0000	0.0018	0.0018	0.0000
2	250	700	0.0000	0.0041	0.0041	0.0081
0	350	800	0.0001	0.0082	0.0081	0.0000
0	450	900	0.0004	0.0151	0.0147	0.0000
2	550	1000	0.0012	0.0260	0.0249	0.0497
2	650	1100	0.0028	0.0424	0.0397	0.0794
10	750	1200	0.0058	0.0659	0.0605	0.6046
26	850	1300	0.0112	0.0984	0.0882	2.2939
27	950	1400	0.0199	0.1415	0.1241	3.3500
22	1050	1500	0.0334	0.1965	0.1688	3.7130
24	1150	1600	0.0532	0.2640	0.2227	5.3445
24	1250	1700	0.0810	0.3434	0.2856	6.8540
11	1350	1800	0.1186	0.4328	0.3565	3.9218
11	1450	1900	0.1675	0.5286	0.4338	4.7719
20	1550	2000	0.2287	0.6259	0.5149	10.2990
8	1650	2100	0.3023	0.7188	0.5969	4.7752
4	1750	2200	0.3871	0.8016	0.6763	2.7052
2	1850	2300	0.4802	0.8700	0.7499	1.4998
3	1950	2400	0.5774	0.9218	0.8149	2.4446
3	2050	2500	0.6732	0.9573	0.8693	2.6080
1	2150	2600	0.7618	0.9792	0.9125	0.9125
						Sum = 56.2352

TABLE 3.6. BLEED SYSTEM RISK AFTER 18 MONTHS

(Utilization Rate = 20 Hours Per Month)

Number of Units (N)	*Current Time on Each Unit (t)*	*Time on Each Unit in 18 Months (t+u)*	*F(t)*	*F(t+u)*	*Each Unit's Risk* $\frac{F(t+u)-F(t)}{1-F(t)}$	*Total Risk* $\frac{F(t+u)-F(t)}{1-F(t)} \cdot N$
0	50	410	0.0000	0.0002	0.0002	0.0000
0	150	510	0.0000	0.0008	0.0008	0.0000
2	250	610	0.0000	0.0020	0.0020	0.0039
0	350	710	0.0001	0.0044	0.0043	0.0000
0	450	810	0.0004	0.0087	0.0083	0.0000
2	550	910	0.0012	0.0160	0.0148	0.0296
2	650	1010	0.0028	0.0273	0.0246	0.0493
10	750	1110	0.0058	0.0444	0.0388	0.3877
26	850	1210	0.0112	0.0688	0.0582	1.5136
27	950	1310	0.0199	0.1022	0.0840	2.2676
22	1050	1410	0.0334	0.1465	0.1170	2.5739
24	1150	1510	0.0532	0.2027	0.1580	3.7909
24	1250	1610	0.0810	0.2714	0.2072	4.9738
11	1350	1710	0.1186	0.3520	0.2648	2.9127
11	1450	1810	0.1675	0.4422	0.3300	3.6296
20	1550	1910	0.2287	0.5334	0.4015	8.0301
8	1650	2010	0.3023	0.6355	0.4775	3.8201
4	1750	2110	0.3871	0.7276	0.5556	2.2222
2	1850	2210	0.4802	0.8091	0.6328	1.2656
3	1950	2310	0.5774	0.8759	0.7064	2.1192
3	2050	2410	0.6732	0.9260	0.7736	2.3208
1	2150	2510	0.7613	0.9660	0.8323	0.8323
						Sum = 42.7432

3.12 CASE STUDY 3: SYSTEM RISK ANALYSIS UTILIZING A SIMULATION MODEL

Assume a jet engine has four independent failure modes:

- Overtemperature
- Vane and Case cracking
- Oil Tube cracking
- Combustion chamber cracking.

The failure distribution of each of these modes is illustrated in Figure 3.9. In addition, there is a scheduled inspection at 1000 hours. At failure or scheduled inspection the modes are made "good-as-new."

1) How many failures can be expected in each mode over the next 2 years? (Assuming a usage rate of 25 hours/month)

2) How will lengthening the inspection interval to 1200 hours change this risk?

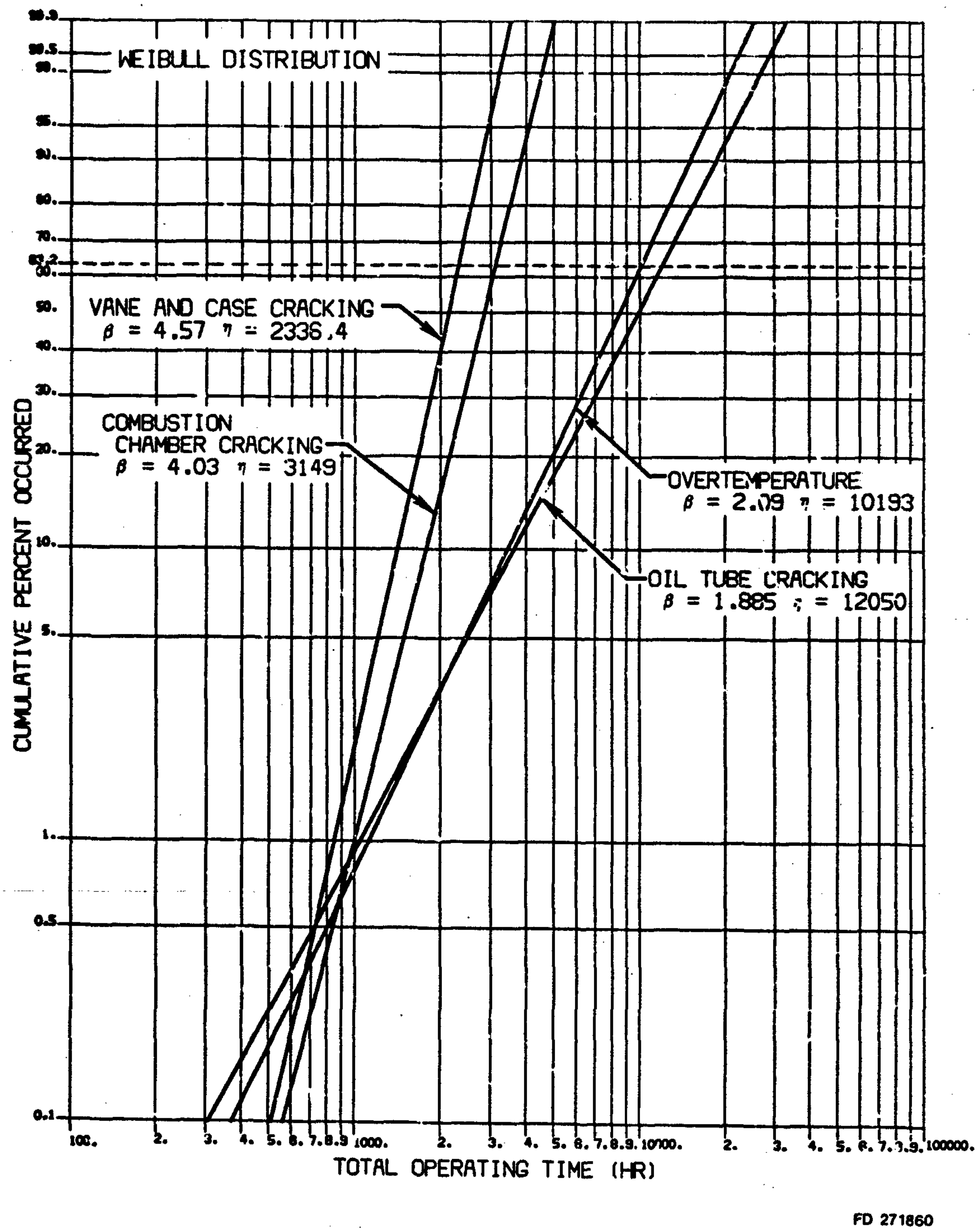


Figure 3.9. Failure Distribution Input to Simulation

There is no easy solution to this problem without simulation. A Monte Carlo simulation based on these groundrules is illustrated in Figure 3.10.

To provide more detail, one engine starting with 0 hours, will be followed step by step to the first scheduled inspection at 1000 hours.

Step 1

Generate random times to failure for each failure mode. First, using a table of random numbers, (Reference 1), four random numbers converted to the 0 to 1 range are 0.007, 0.028, 0.517, and 0.603.

Using Equation 3.4:

F_1 = Overtemperature $= 10{,}193\left[\ln\left(\frac{1}{1-0.007}\right)\right]^{1/2.09} = 951$ hours

F_2 = Vane and case cracking $= 2{,}336\left[\ln\left(\frac{1}{1-0.028}\right)\right]^{1/4.57} = 1{,}072$ hours

F_3 = Oil tube cracking $= 12{,}050\left[\ln\left(\frac{1}{1-0.517}\right)\right]^{1/1.885} = 10{,}180$ hours

F_4 = Combustion chamber cracking $= 3{,}149\left[\ln\left(\frac{1}{1-0.603}\right)\right]^{1/4.03} = 3{,}088$ hours

Steps 2 & 3

The minimum of the times-to-failure and inspection time is 951 hours; therefore, the scheduled inspection was not reached.

Step 4

This failure was an overtemperature (F_1) and is recorded as occurring 951/(25 hours usage) 38 months in the future.

Generate another time to failure for F_1, using the next random number, 0.442.

$$\text{New } F_1 = 10{,}193\left[\ln\left(\frac{1}{1-0.442}\right)\right]^{1/2.09} = 7{,}876 \text{ hours}$$

+ 951 hours on F failure

8,827 hours

Ref. 1, *A Million Random Digits With 100,000 Normal Deviates*, The Free Press; Rand Corporation, 1955.

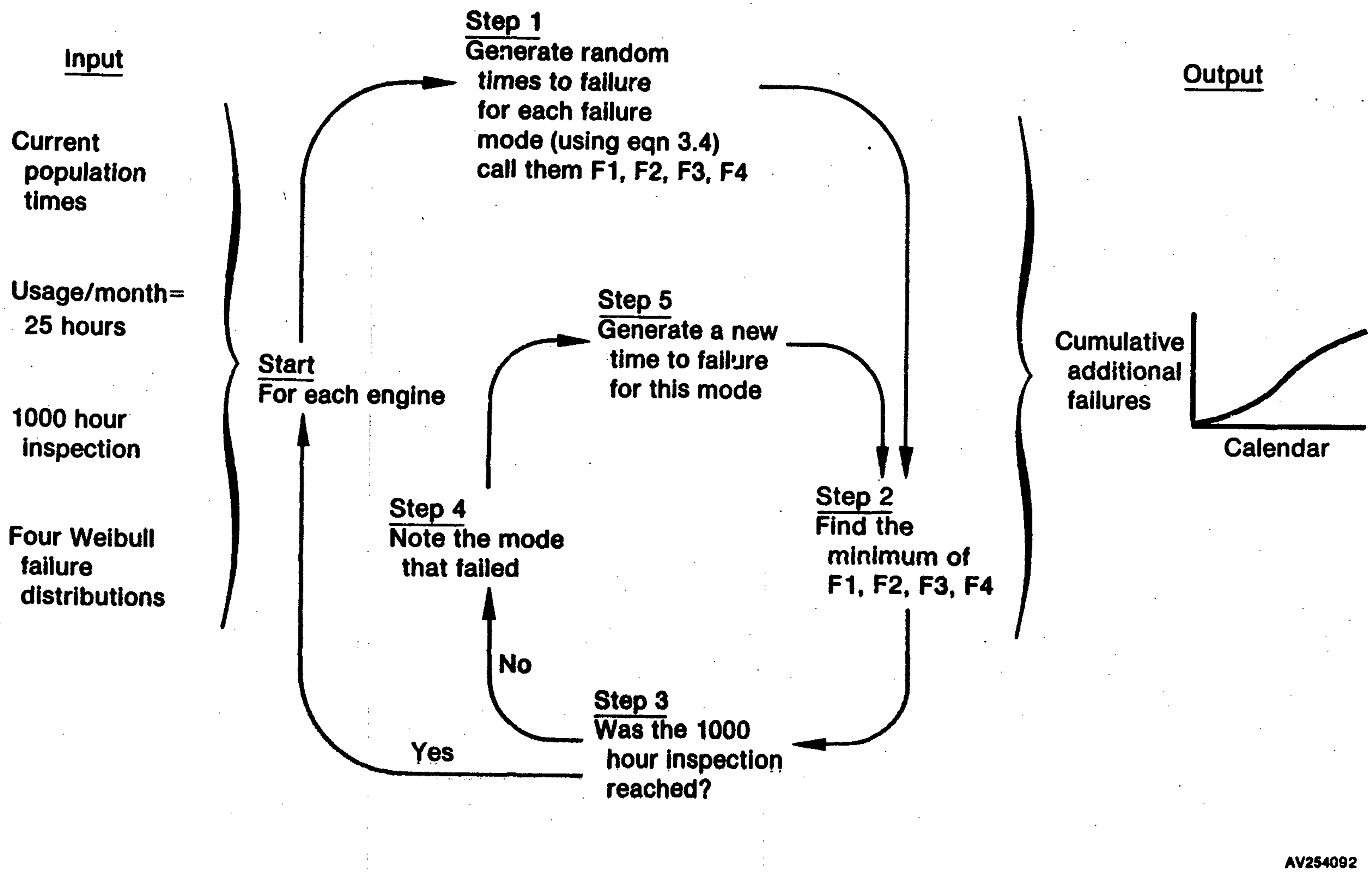


Figure 3.10. Risk Analysis Simulation Outline

Now, the minimum of (F_1, F_2, F_3, F_4, 1000 hours) is 1000 hours, which is the scheduled inspection. This process can be continued for as many inspection intervals as desired.

For engines with greater than zero hours initially, the Monte Carlo process must be modified. First, the time since last 1000-hour inspection is calculated and used as the engine's initial age (since engines are made "good as new" at each 1000-hour inspection). Then, note that the first set of four random failure times must be greater than the engine's initial age (since all of the engines in the histogram are suspensions). If any are less, other random numbers are drawn until all four failure times are greater than the initial age.

The above procedure is followed for each engine in the population (Figure 3.11) and is repeated several times so that an average risk can be calculated.

The simulation in Figure 3.10 was run, and the risk for the first 24 months is presented in Table 3.7 for the 1000 hour inspection, and in Table 3.8 for the 1200 hour inspection. A plot comparing the two risks is presented in Figure 3.12. Increasing the inspection interval to 1200 hours increases the expected number of failures from 25 to 34, a delta of 9, by the end of 1981.

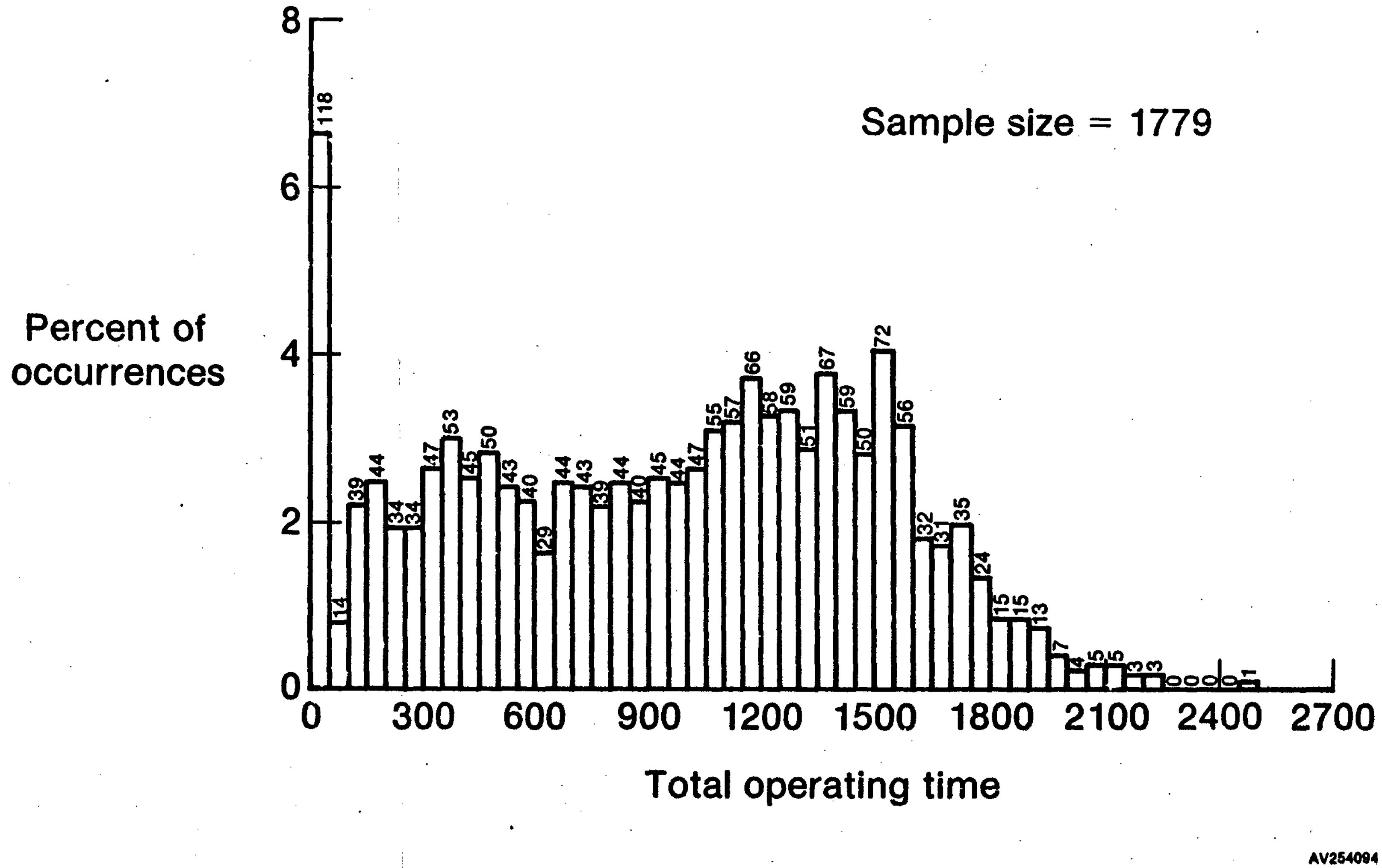


Figure 3.11. Module Population

TABLE 3.7. SIMULATION OUTPUT FOR 1000 HOUR INSPECTION

			Cumulative Incidents			
Month	*EFH**	*Cum EFH**	*Oil Tube*	*Vane Case*	*Over/Temp*	*Comb. Chamber*
******** 1980 ********						
Jan	29,225	29,225	0.00	0.00	0.00	0.00
Feb	29,225	58,450	0.17	0.33	0.21	0.17
Mar	29,225	87,675	0.38	0.67	0.47	0.34
Apr	29,225	116,900	0.60	1.15	0.74	0.58
May	29,225	146,125	0.78	1.47	0.95	0.74
Jun	29,225	175,350	0.92	1.71	1.13	0.87
Jul	29,225	204,575	1.22	2.27	1.49	1.15
Aug	29,225	233,800	1.46	2.81	1.77	1.41
Sep	29,225	263,025	1.66	3.16	2.02	1.60
Oct	29,225	292,250	1.95	3.90	2.36	1.95
Nov	29,225	321,474	2.07	4.03	2.51	2.03
Dec	29,225	350,699	2.38	4.90	2.87	2.45
******** 1981 ********						
Jan	29,225	379,924	2.66	5.55	3.19	2.77
Feb	29,225	409,149	2.77	5.83	3.32	2.91
Mar	29,225	438,374	2.87	6.13	3.44	3.05
Apr	29,225	467,599	3.07	6.68	3.67	3.31
May	29,225	496,824	3.28	7.33	3.91	3.62
Jun	29,225	526,049	3.37	7.48	4.02	3.69
Jul	29,225	555,274	3.64	8.26	4.33	4.06
Aug	29,225	584,499	3.70	8.45	4.40	4.15
Sep	29,225	613,724	3.76	8.59	4.47	4.21
Oct	29,225	642,949	3.80	8.59	4.47	4.21
Nov	29,225	672,174	4.16	9.40	4.95	4.62
Dec	29,225	701,399	4.44	9.96	5.29	4.90

*EFH = engine flight hours

TABLE 3.8. SIMULATION OUTPUT FOR 1200 HOUR INSPECTION

			Cumulative Incidents			
Month	*EFH**	*Cum EFH**	*Oil Tube*	*Vane Case*	*Over/Temp*	*Comb. Chamber*
* * * * * * * * 1980 * * * * * * * *						
Jan	29,225	29,225	0.00	0.00	0.00	0.00
Feb	29,225	58,450	0.17	0.53	0.21	0.24
Mar	29,225	87,675	0.40	1.28	0.47	0.58
Apr	29,225	116,900	0.69	2.44	0.79	1.08
May	29,225	146,125	0.91	3.21	1.05	1.43
Jun	29,225	175,350	1.21	4.02	1.40	1.81
Jul	29,225	204,575	1.40	4.86	1.62	2.17
Aug	29,225	233,800	1.57	5.25	1.82	2.36
Sep	29,225	263,025	1.84	6.26	2.13	2.81
Oct	29,225	292,250	1.93	6.64	2.43	2.98
Nov	29,225	321,474	2.12	7.39	2.55	3.31
Dec	29,225	350,699	2.35	8.25	3.08	3.69
* * * * * * * * 1981 * * * * * * * *						
Jan	29,225	379,924	2.61	9.26	3.21	4.14
Feb	29,225	409,149	2.72	9.73	3.34	4.34
Mar	29,225	438,374	3.0[illegible]	10.99	3.46	4.89
Apr	29,225	467,599	3.31	12.16	3.79	5.40
May	29,225	496,824	3.31	12.16	3.93	5.40
Jun	29,225	526,049	3.65	13.72	4.04	6.07
Jul	29,225	555,274	3.65	13.72	4.35	6.07
Aug	29,225	584,499	3.93	14.92	4.42	6.58
Sep	29,225	613,724	3.96	14.92	4.59	6.58
Oct	29,225	642,949	4.01	15.16	4.64	6.68
Nov	29,225	672,174	4.16	15.77	4.99	6.94
Dec	29,225	701,399	4.46	16.47	5.32	6.94

*EFH = engine flight hours

3.13 PROBLEMS

Problem 3-1

A fleet of 100 engines is subjected to a Weibull failure mode. The Weibull has a slope of 3 and a characteristic life of 1000 hours. The current engine times are as follows:

Number of Engines	*Engine Time*
20	150 hrs
20	200
20	250
20	300
20	350

A.) What is the expected number of failures now? B.) How many additional engines will be expected to fail in 6 months if the utilization rate is 25 hr/mo? Assume that failed units are not fixed.

Problem 3-2

A turbine airfoil has caused unscheduled engine removals at the following times and locations.

Time at Failure	*Location*
684 (hours)	A
821	A
812	A
701	A
770	A
845	A
855	B
850	C
806	E
756	G
755	H
741	G
681	E
667	C
649	B
603	B
600	C
596	G
576	D
504	E
476	H

A) Generate a Weibull using the attached populations, overall (Figure 3.13) and at Location A (Figure 3.14). How do these Weibulls compare?

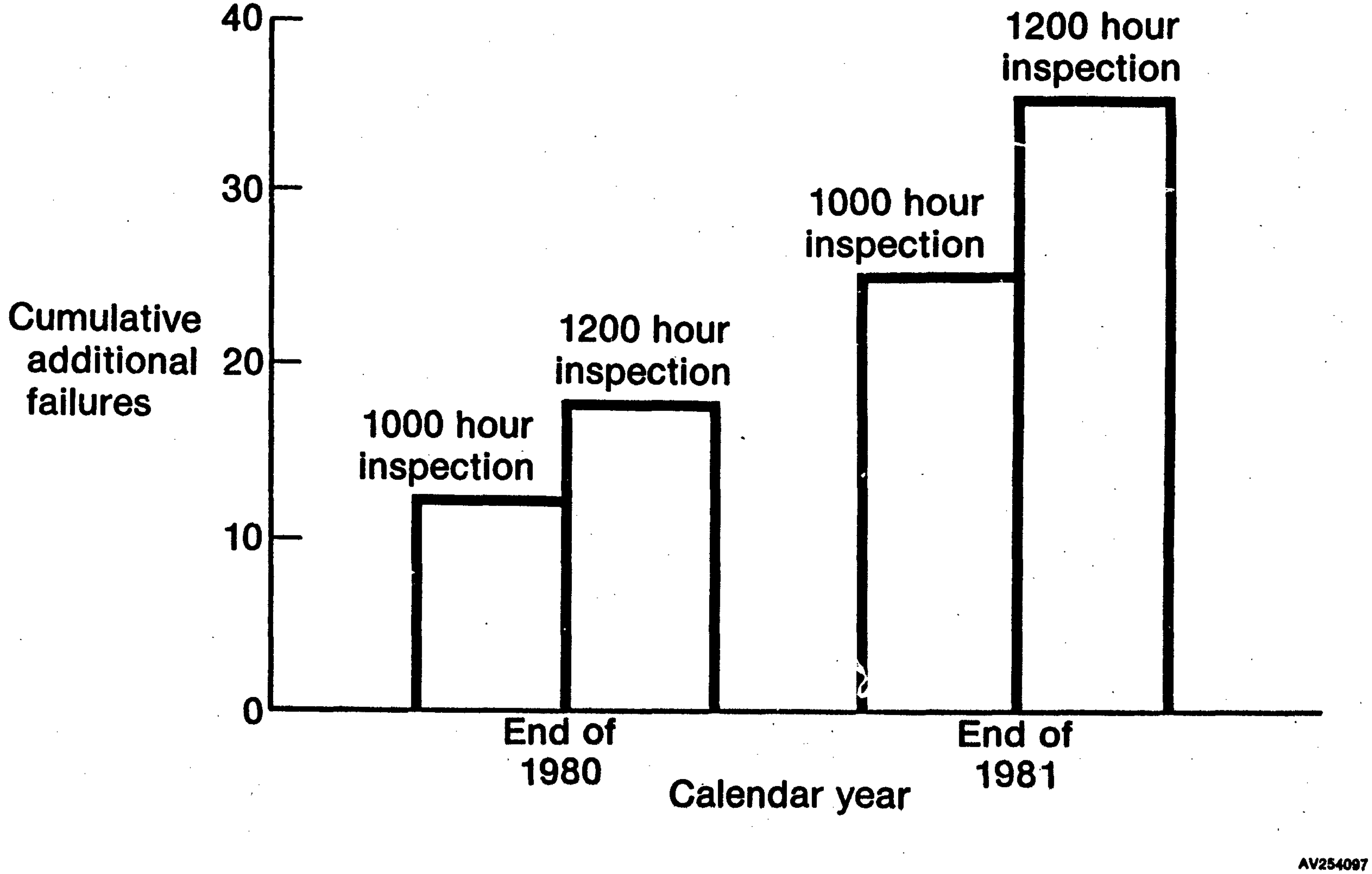


Figure 3.12. Risk Analysis Comparison

B) How many failures can be expected in the next 12 months?, the next 24 months? from each population? (Use 30 hours/mo.)

Problem 3-3

Given a control failure mode with $\beta = 1.26$ and $\eta = 19{,}735$ total operating hours, and the population of nonfailed units Figure 3.15, A) how many failures can be expected by the time each unit has reached 1000 hours? B) 2000 hours? C) If the life of a control is 4000 hours, what is the projected total number of failures in the life of the control if no further controls are added to the population? D) If inspections "zero-time," or make the control units "good-as-new," how many failures are projected to occur in this population by 4000 hours with a 1000 hour inspection? E) with a 2000 hour inspection?

Problem 3-4

Using the table of 0-1 random numbers in Table 3.9, and the three Weibull failure modes:

a. $\beta = 0.76$
$\eta = 96{,}587$ hours

b. $\beta = 2.638$
$\eta = 4996.$ hours

c. $\beta = 7.4$
$\eta = 1126.$ hours

Assume two scheduled inspections, at 1000 hours and 2000 hours, that make modes a and c "good-as-new," while not helping mode b. A usage rate of 25 hours per month is assumed.

The following population of 5 engines is at risk:

1 engine at 100 hours, 1 engine at 200 hours, 1 engine at 500 hours,

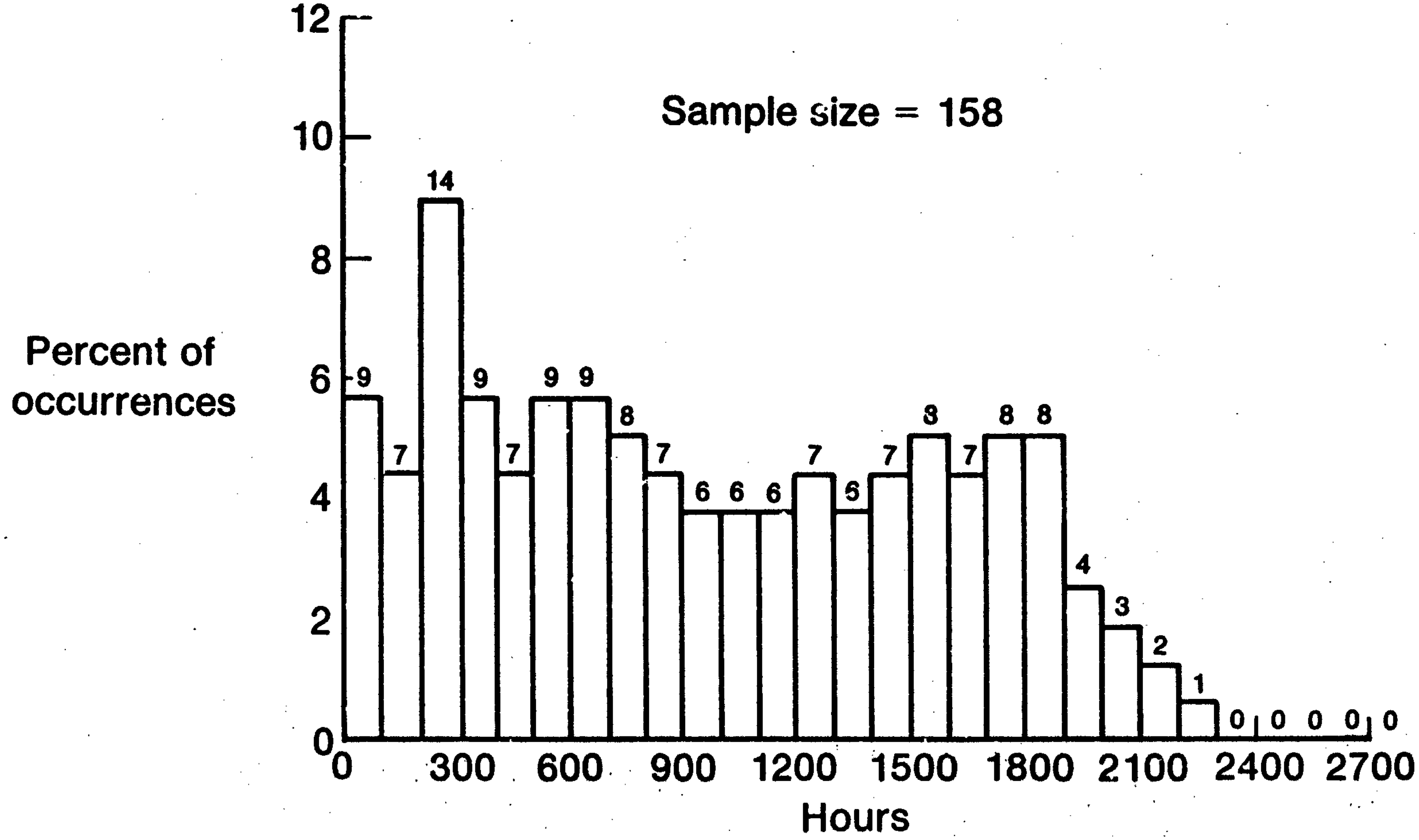
1 engine at 700 hours, and 1 engine at 900 hours.

A) How many failures will occur over the next 48 months?

Use the Monte Carlo simulation technique to solve this problem.

B) Would it be advisable to drop the 1000 hour inspection?

Solutions to these problems are in Appendix J.



AV254098 823011 1226B

Figure 3.13. Overall Population

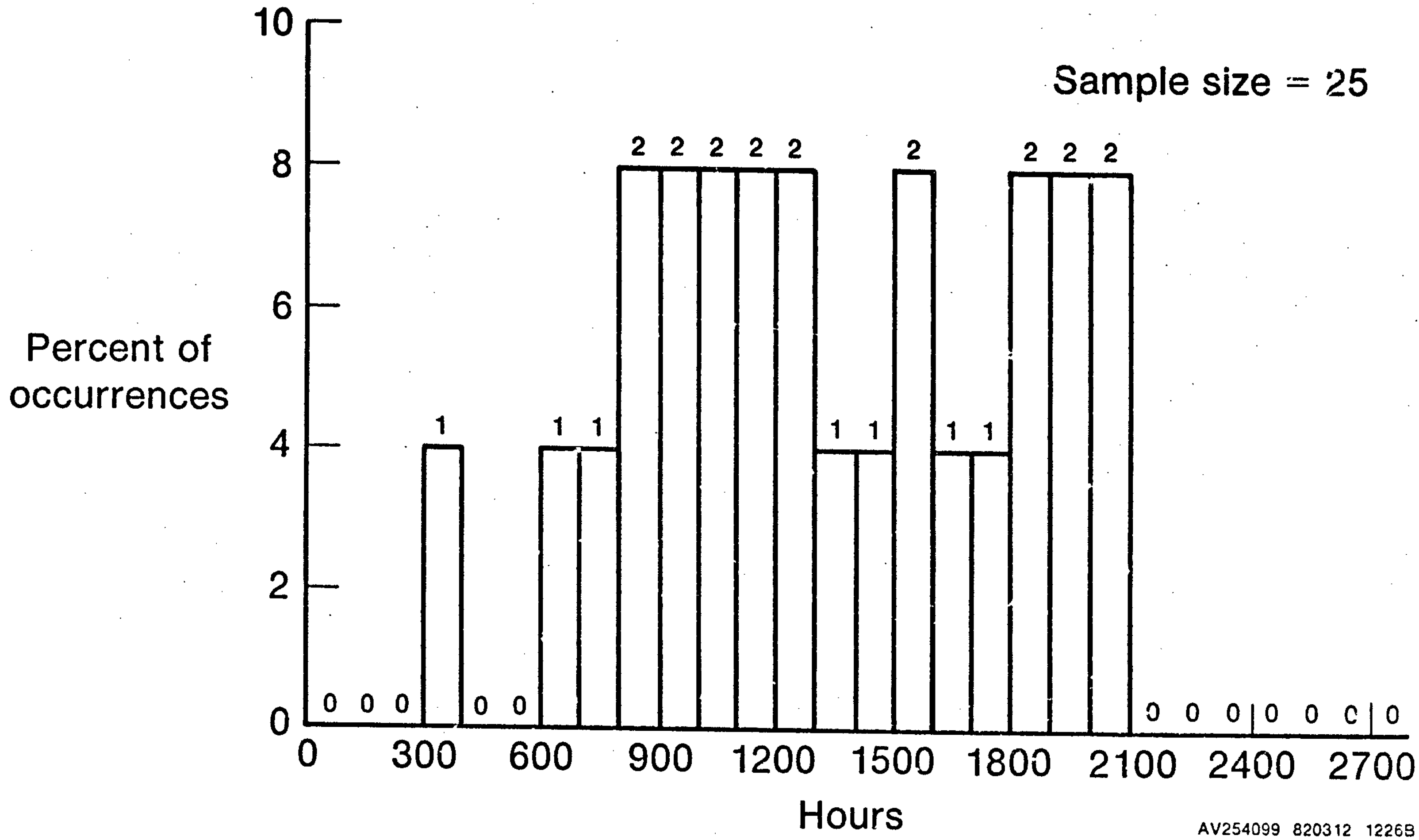


Figure 3.14. Location A Population

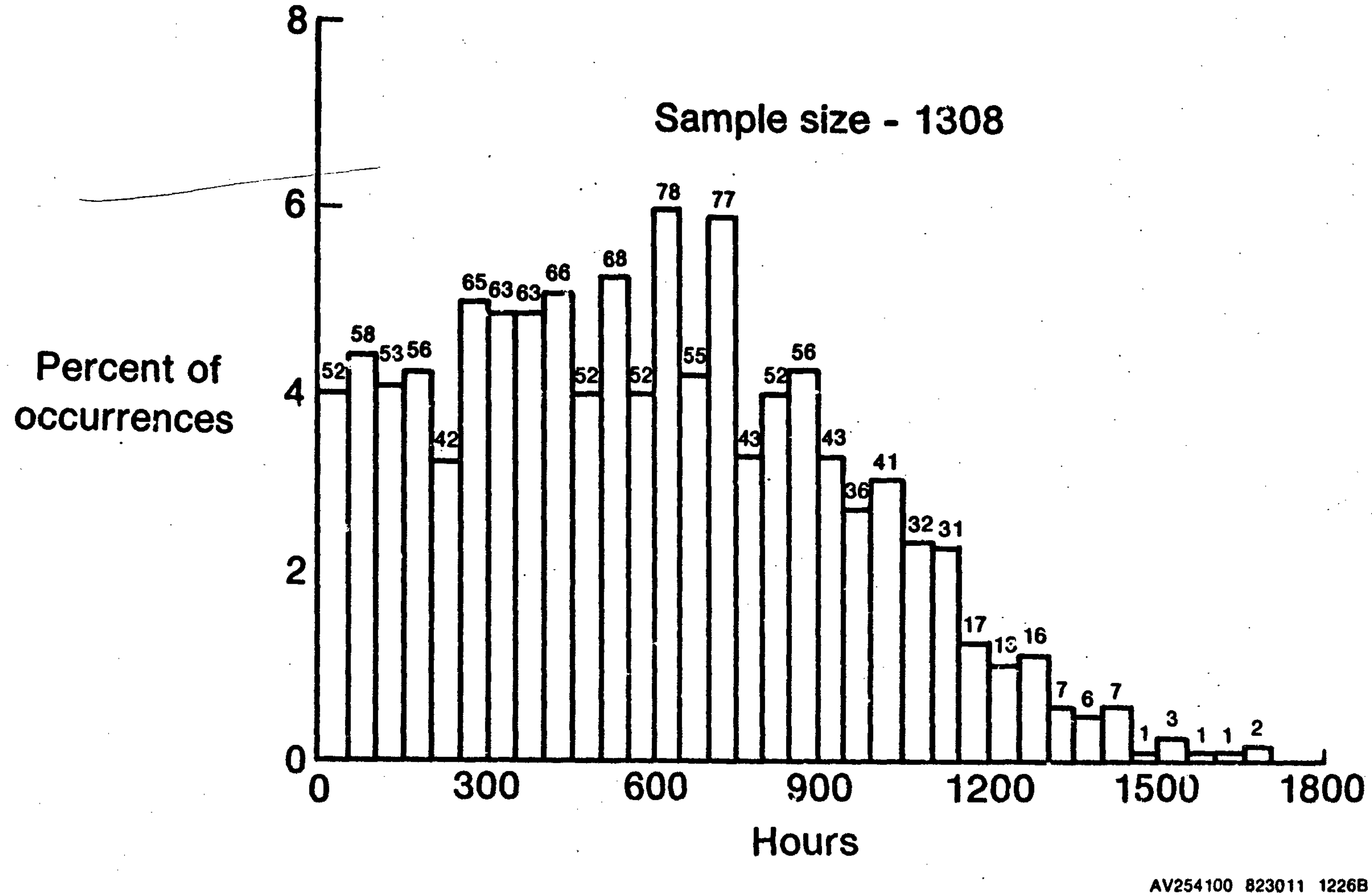


Figure 3.15. Control Population

TABLE 3.9. TABLE OF UNIFORM RANDOM NUMBERS FROM 0. TO 1.0

0.329	0.604	0.615	0.300	0.070	0.845	0.494	0.624	0.085	0.194
0.612	0.337	0.393	0.163	0.774	0.620	0.596	0.503	0.857	0.794
0.545	0.945	0.357	0.429	0.769	0.675	0.689	0.203	0.643	0.577
0.232	0.511	0.311	0.213	0.124	0.827	0.354	0.556	0.811	0.811
0.221	0.480	0.345	0.167	0.390	0.987	0.428	0.257	0.298	0.198
0.210	0.457	0.010	0.083	0.837	0.265	0.638	0.943	0.747	0.164
0.519	0.668	0.717	0.230	0.133	0.672	0.658	0.491	0.772	0.676
0.166	0.037	0.971	0.169	0.815	0.876	0.668	0.649	0.205	0.551
0.138	0.601	0.761	0.490	0.655	0.238	0.277	0.123	0.918	0.984
0.214	0.738	0.224	0.706	0.748	0.090	0.389	0.699	0.562	0.761
0.418	0.422	0.402	0.270	0.928	0.982	0.365	0.933	0.323	0.367
0.950	0.469	0.709	0.431	0.854	0.363	0.574	0.630	0.521	0.974
0.202	0.503	0.434	0.394	0.851	0.909	0.168	0.058	0.673	0.012
0.180	0.104	0.384	0.013	0.364	0.480	0.687	0.636	0.340	0.805
0.447	0.360	0.506	0.980	0.605	0.408	0.833	0.544	0.861	0.476
0.412	0.785	0.084	0.222	0.750	0.600	0.495	0.497	0.821	0.105
0.580	0.332	0.855	0.990	0.765	0.669	0.895	0.635	0.842	0.850
0.083	0.963	0.134	0.847	0.717	0.054	0.420	0.249	0.041	0.502
0.609	0.996	0.793	0.526	0.159	0.861	0.507	0.826	0.249	0.688
0.551	0.198	0.701	0.376	0.932	0.888	0.655	0.608	0.838	0.703

CHAPTER 4

WEIBAYES — WHEN WEIBULLS ARE IMPOSSIBLE

4.1 FOREWORD

At times a Weibull plot cannot be made because of deficiencies in the data. Typical situations would be when:

(1) There are too few or no failures.
(2) The age of the units is unknown, and only the number of failures is known.
(3) A test plan for a new design is needed.

Weibayes analysis has been developed to solve problems when Weibull analysis cannot be used. Weibayes is never preferred over Weibull analysis but is often required because of weaknesses in the data. Weibayes is defined as Weibull analysis with an assumed β parameter. Since the assumption requires judgment, this analysis is regarded as an informal Bayesian procedure.

4.2 WEIBAYES METHOD

In a Weibayes analysis, the slope/shape parameter β is assumed from historical failure data or from engineering knowledge of the physics of the failure. Depending upon the situation, this may be a strong or weak assumption. Given β, an equation may be derived (Appendix E) using the method of maximum likelihood to determine the characteristic life, η.

$$\eta = \left[\sum_{i=1}^{N} \frac{t_i^{\beta}}{r}\right]^{1/\beta} \tag{4.1}$$

Where t_i is the time or cycles on unit$_i$, r is the number of failed units and η is the maximum likelihood estimate of the characteristic life.

With β assumed and η calculated from equation (4.1), a Weibull equation is determined. A Weibayes line can be plotted on Weibull paper. The plot is used exactly like a Weibull distribution.

4.3 WEIBAYES — NO FAILURES

In many Weibayes problems no failure has occurred. In this case, a second assumption is required. The first failure is assumed to be imminent; i.e. $r = 1.0$ (otherwise, the denominator in equation (4.1) would be zero). The Weibayes line based on assuming one failure is conservative, with at least 63% confidence that the true Weibull lies to the right of the Weibayes line. (See Appendix E.)

The exact confidence level of the Weibayes lower bound is unknown because it depends on the time to the first failure. If the Weibayes line is always constructed immediately before the first failure, the Weibayes confidence level is 63%. If Weibayes analyses are consistently done long before the first failure, the confidence level is actually much higher than 63%. Therein, Weibayes displays conservatism since the confidence level, while unknown, is at least 63%.

4.4 WEIBEST — NO FAILURES

In the early development and use of this analysis* 0.693 failures would be assumed instead of 1.0. This is less conservative. The result was called a Weibest line. The Weibest line is a 50% lower confidence bound on the true Weibull characteristic life versus 63% for Weibayes with r = 1.0. In fact, Weibayes lines may be calculated for any confidence level (Appendix E).

4.5 UNKNOWN FAILURE TIMES

Sometimes the number of failures is known, but not the times to failure; again Weibayes may provide a solution. For example, if the failed part is nonserialized and the component or system has been through overhaul, it may be impossible to determine the time on the failed unit(s) or the success unit(s). However, if the time on the components or systems is known, it may be reasonable to assume that the same distribution of times applies to the nonserialized parts. In this application, there is more uncertainty in assuming a value for β. If the physics of failure are known, a library of Weibull failure modes may provide an estimate or a range of estimates; the maximum and minimum β may each be used to determine the sensitivity of the analysis to the assumption.

If the times on the failed units are known but the times on the successful units are unknown, a Weibull shift method may be employed. (See Section 6.3.)

4.6 WEIBAYES WORRIES AND CONCERNS

The Weibayes method is required when there are deficiencies in the data or when the data are not available. The Weibull method is always preferred over Weibayes, so it is appropriate to critically question the assumptions required by the Weibayes method in each case since the answers to these questions vary for each application. Of course, the validity of the results depends on the validity of the assumptions. Typical questions to be raised are:

(1) How valid is the assumed slope, β? If this assumption is shaky, should a range of slopes be tried?

(2) With a redesign, what is the probability that a new failure mode is present? A Weibayes test may not discover a new mode.

(3) With nonserialized parts, some assumption must be made to obtain success or failure times. How valid is the assumption?

4.7 EXAMPLES OF PROBLEMS/ANALYTICAL SOLUTIONS

Problem 1) Fifteen vane and case failures have been experienced in a large fleet of engines. Weibull analysis provides a β of 5.0 (see Figure 4.1). Three redesigned compressor cases have been tested in engines to 1600, 2900 and 3100 hours without failure. Is this enough testing to substantiate the redesign?

* Mr. Joseph W. Grdenick of Pratt & Whitney Aircraft/Commercial Products Division is credited for much of the original development of the Weibest concept

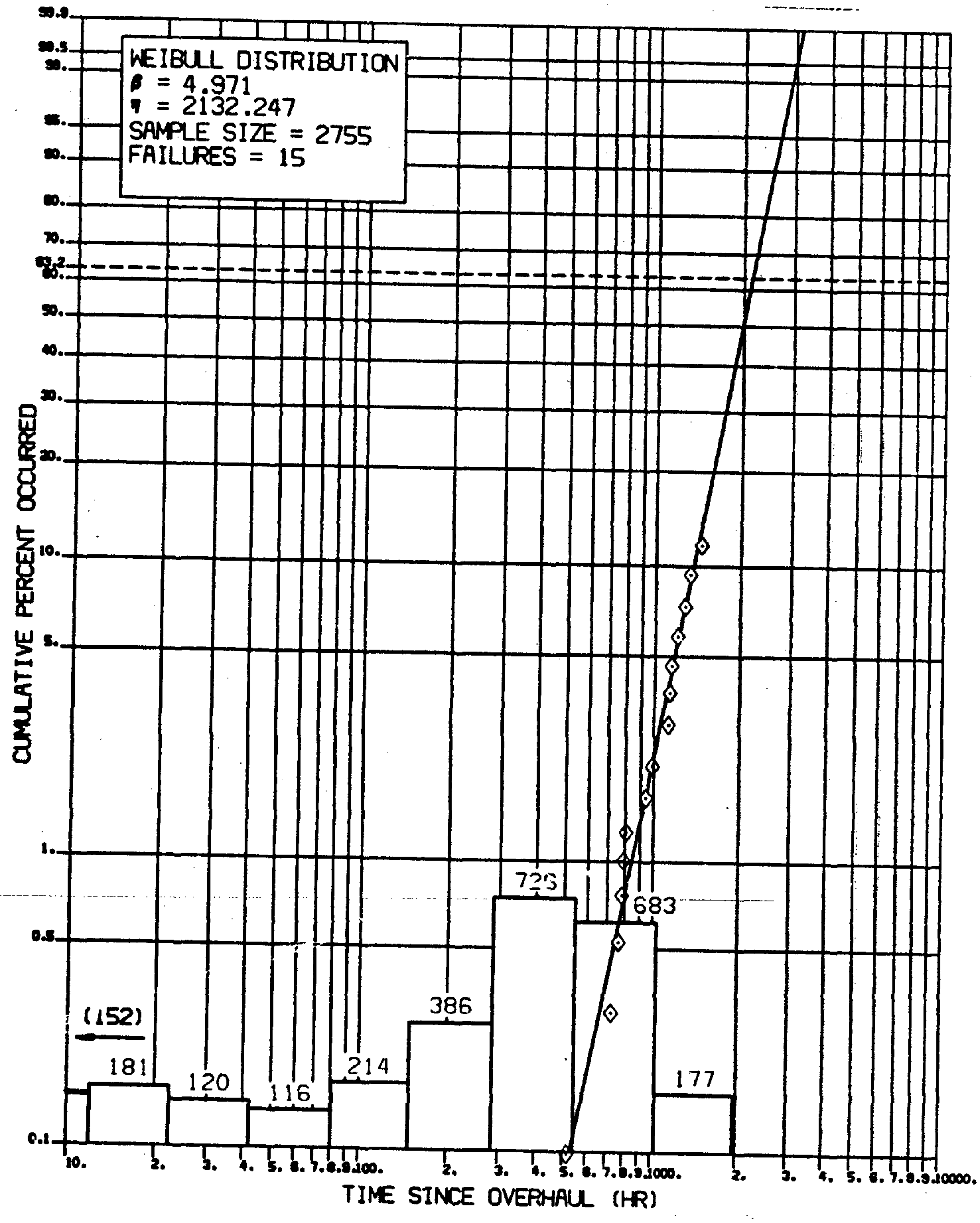


Figure 4.1. Compressor Vane and Case

Assuming $\beta = 5.0$ and given the times on the three redesigned units, equation (4.1) may be used to calculate the characteristic life for a Weibayes solution.

$$\eta = \left[\frac{(1600)^5 + (2900)^5 + (3000)^5}{1}\right]^{1/5} = 3406 \text{ hr} \tag{4.2}$$

The Weibayes line is plotted in Figure 4.2. We may state with 63% confidence that the failure mode for the redesigned units is to the right of this line and, therefore, significantly better than the bill-of-material vane and case. It is possible that the redesign has eliminated this failure mode but that cannot be proven with this sample of data. As more time is put on these units without failure, the Weibayes line will move further to the right and more confidence will be gained that the failure mode has been eliminated. The assumption of slope, in this case, is based on an established Weibull failure mode and is valid.

Problem 2) There have been 38 turbopump failures in service (Figure 4.3). Based on the physics of the failure, an accelerated bench test was designed and two more turbopumps failed in a much shorter time (Figure 4.4). Notice that the bench test Weibull has the same slope as the field failure Weibulls. This provides some confidence that the accelerated test provides the same failure mode experienced in service. The turbopump was redesigned to fix the problem and two units were tested on the bench to 500 hours without failure under the same accelerated conditions. Is the redesign successful? What service experience should be expected?

Using equation 4.1 and the slope from the Weibulls in Figure 4.3, the Weibayes characteristic life is calculated, assuming the first failure is imminent.

$$\eta = \left[\frac{500^{2.7} + 500^{2.7}}{1}\right]^{1/2.7} = 646 \text{ hr} \tag{4.3}$$

This Weibayes line is plotted on Figure 4.5. If we assume that the ratio of characteristic lives (η's) for the B/M pump in service to the B/M pump in the rig test is a measure of the acceleration of the test, a Weibayes line can be estimated for the redesigned pump in service. This line is also plotted in Figure 4.5.

$$\eta_{Redesigned/Service} = (\eta_{B/M/SVC} \div \eta_{B/M/Rig})\, \eta_{Redesign/Rig}$$

$$= (2186.2 \text{ hr} \div 140 \text{ hr})\ 646.3 \text{ hr}$$

$$= (15.6)\ 646.3$$

$$\eta_{Redesigned/Service} = 10{,}082 \text{ hr}$$

Problem 3) One batch of turbopumps (DF3) produced nine service failures involving fire in flight. From a Weibull analysis, it was decided to replace these pumps after 175 hours of operation. Two other batches of these pumps, DF1 and DF2, had more service time but no failures. Teardown and inspection of some of these pumps showed that the failure mode (swelling of the ball bearing plastic cage) was present but to a lesser degree. There were not enough spare pumps to immediately replace the DF1 and DF2 units. How long can replacement of DF1 and DF2 be safely delayed?

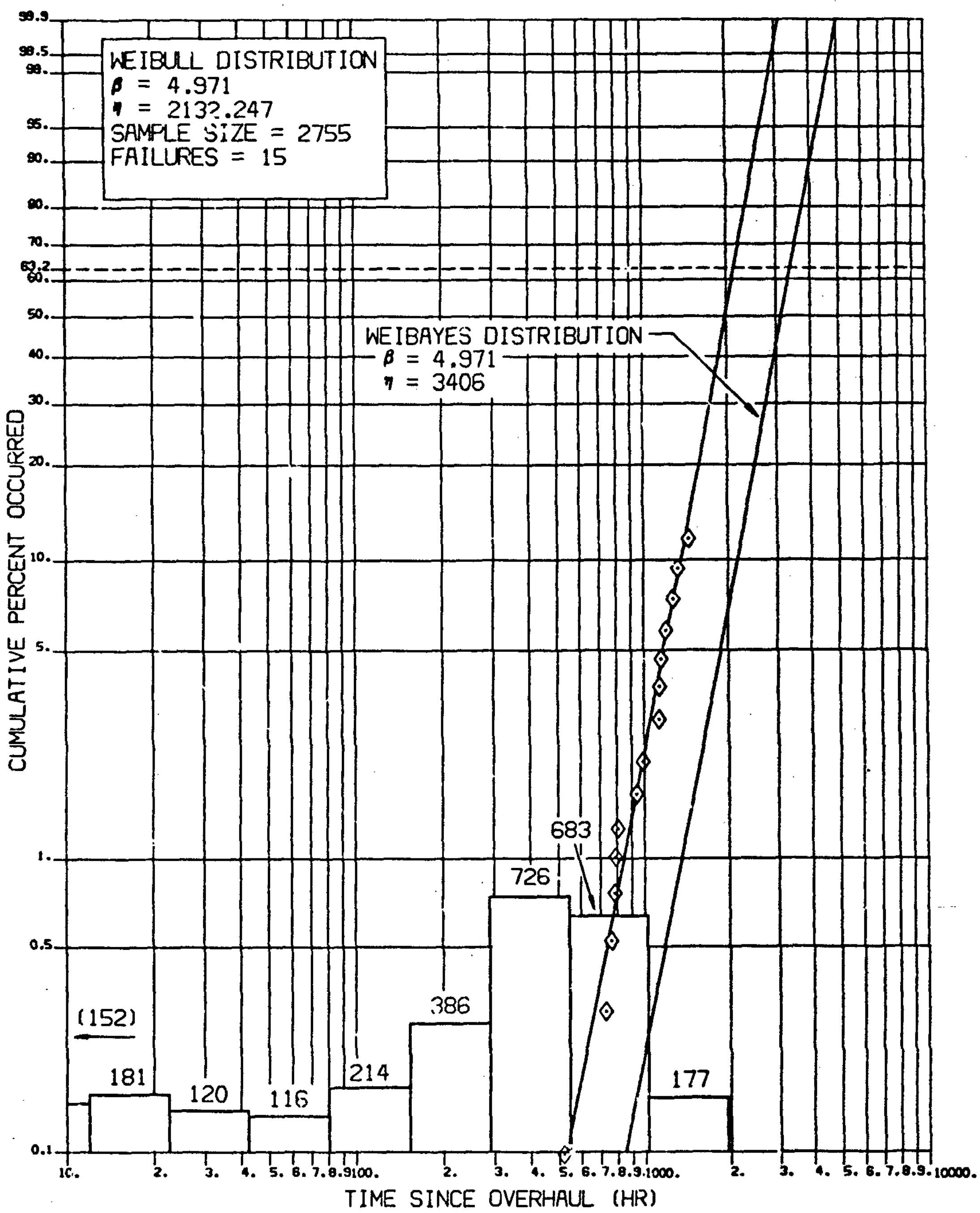


Figure 4.2. Weibayes Evaluation of New Design in Accelera ed Test

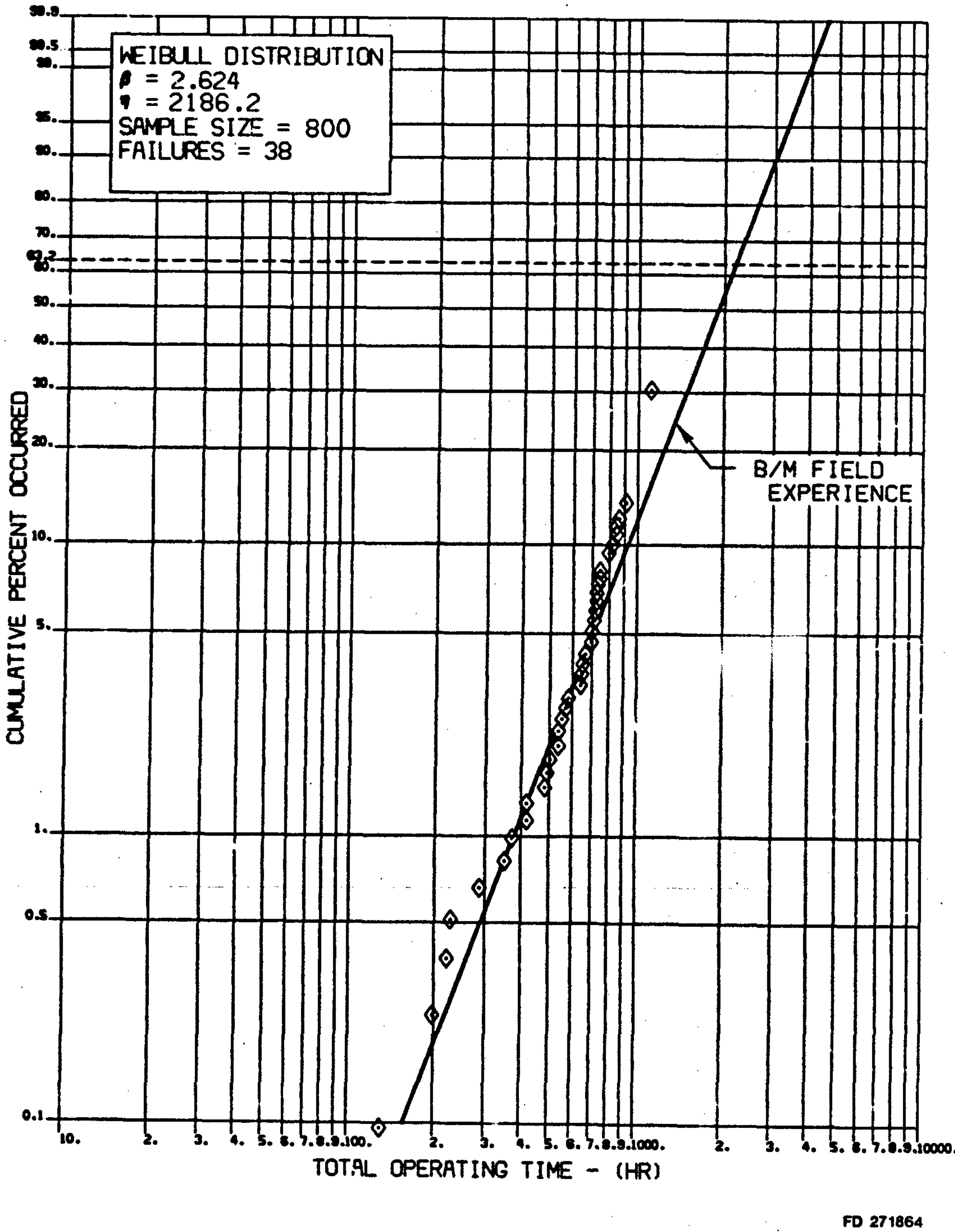


Figure 4.3. Weibull Evaluation of B/M Design

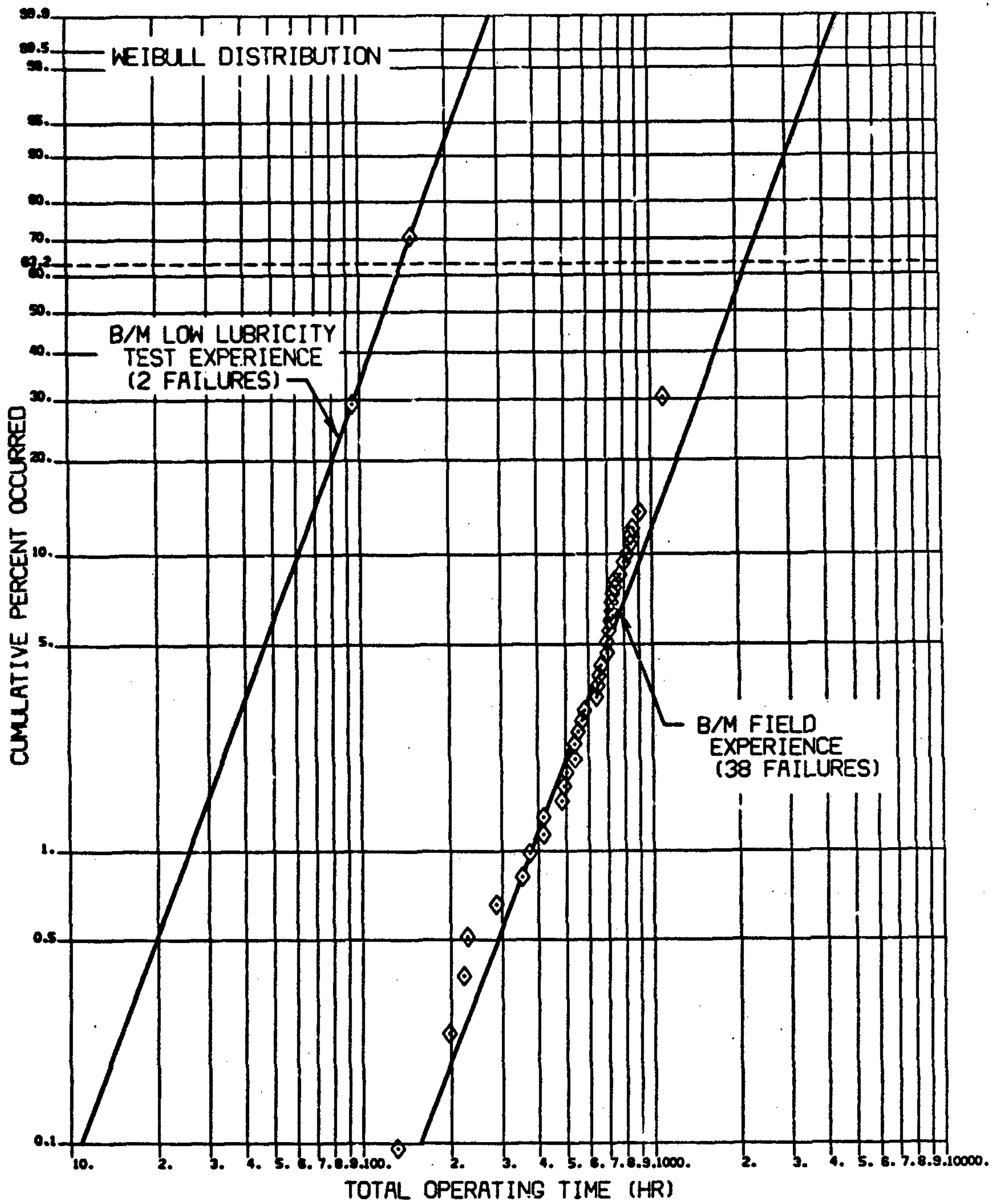


Figure 4.4. Weibayes Evaluation of New Design in Accelerated Test

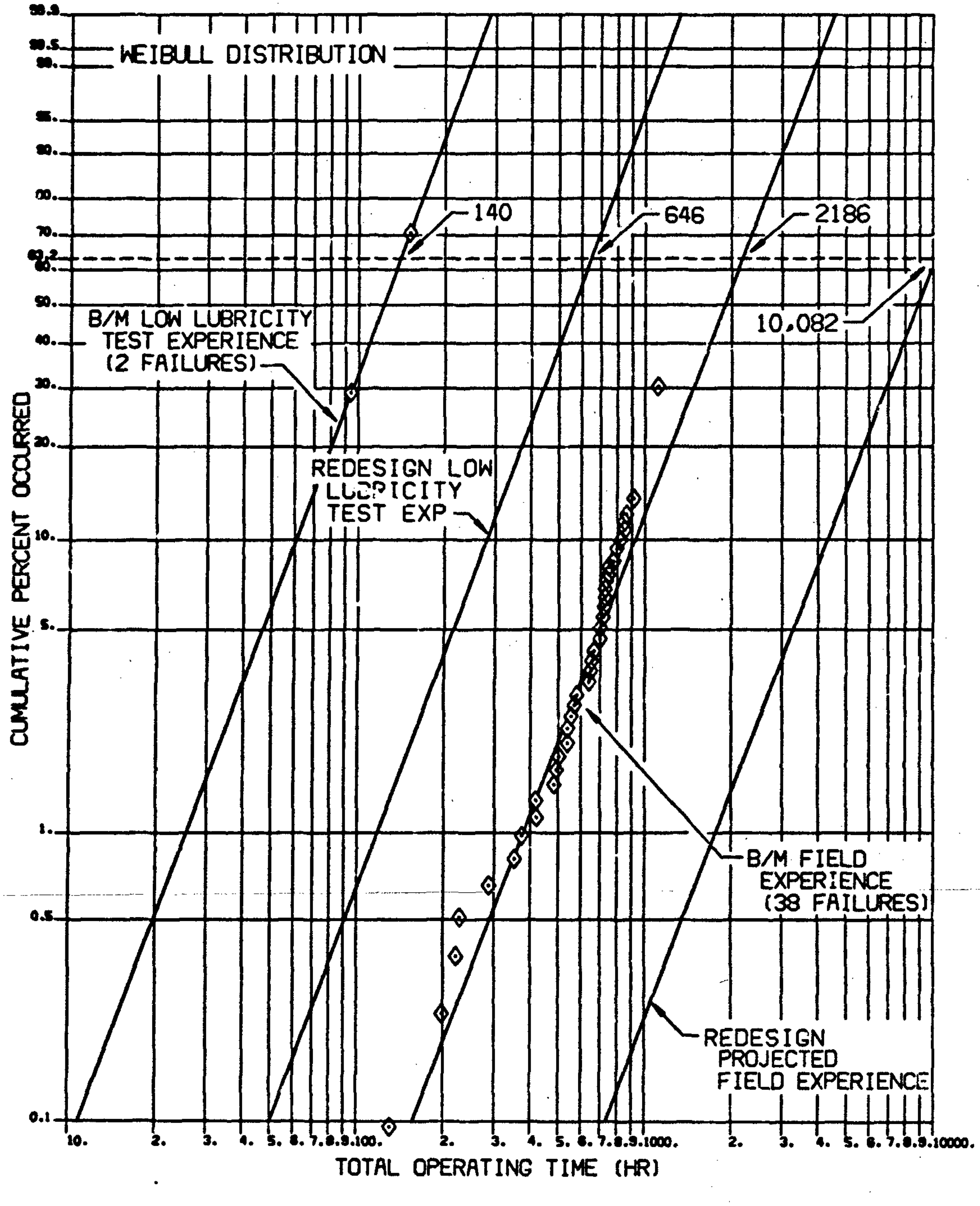


Figure 4.5. Hydraulic Pump Failures

There were no failures in DF1 and DF2 even though symptoms of the failure mode were present. A Weibayes analysis, using the existing Weibull slope of β = 4.6 and assuming the 0.693 failures were imminent, produced the Weibest (50% confidence) line shown in Figure 4.6. The DF3 retrofit at 175 hours corresponds to a risk level of B.7 as indicated in Figure 4.6. The same risk level was applied to the Weibayes line and a 700 hr safe period was recommended. DF1 and DF2 pumps were replaced when they reached 700 hours. This did not create a supportability problem as these pumps had acquired very little time. Weibest and Weibayes lines move to the right with time as long as no failures are observed due to the increase in success time. In this case, the Weibest B.7 time eventually exceeded the pumps' overhaul time of 1000 hours. Therefore, many pumps were utilized to their full life without premature replacement based on the Weibest Analysis.

4.8 PROBLEMS

Problem 4-1

Two bolt failures due to low cycle fatigue have been observed in a flight test fleet of six engines having the following times: 100, 110, 125, 150, 90 and 40 hr. The bolts are not serialized and as the failures were discovered after the engines were overhauled, it is not known which engines had the failed parts. If low cycle fatigue failure modes usually have slope parameters between 2 and 5, and after rebuild the engine will accumulate 100 hours in the next year, predict the number of expected failures. (Assume the two new replacement bolts are installed in the rebuilds of the high time engines.)

Problem 4.2

The design system predicted B.1 life for the compressor disk is 1000 cycles. Five disks have accumulated 1500 cycles and five have 2000 cycles without any failures. If most disk LCF failures have a β of 3.0, is this success data sufficient to increase the predicted design life?

Solutions to these problems are in Appendix J.

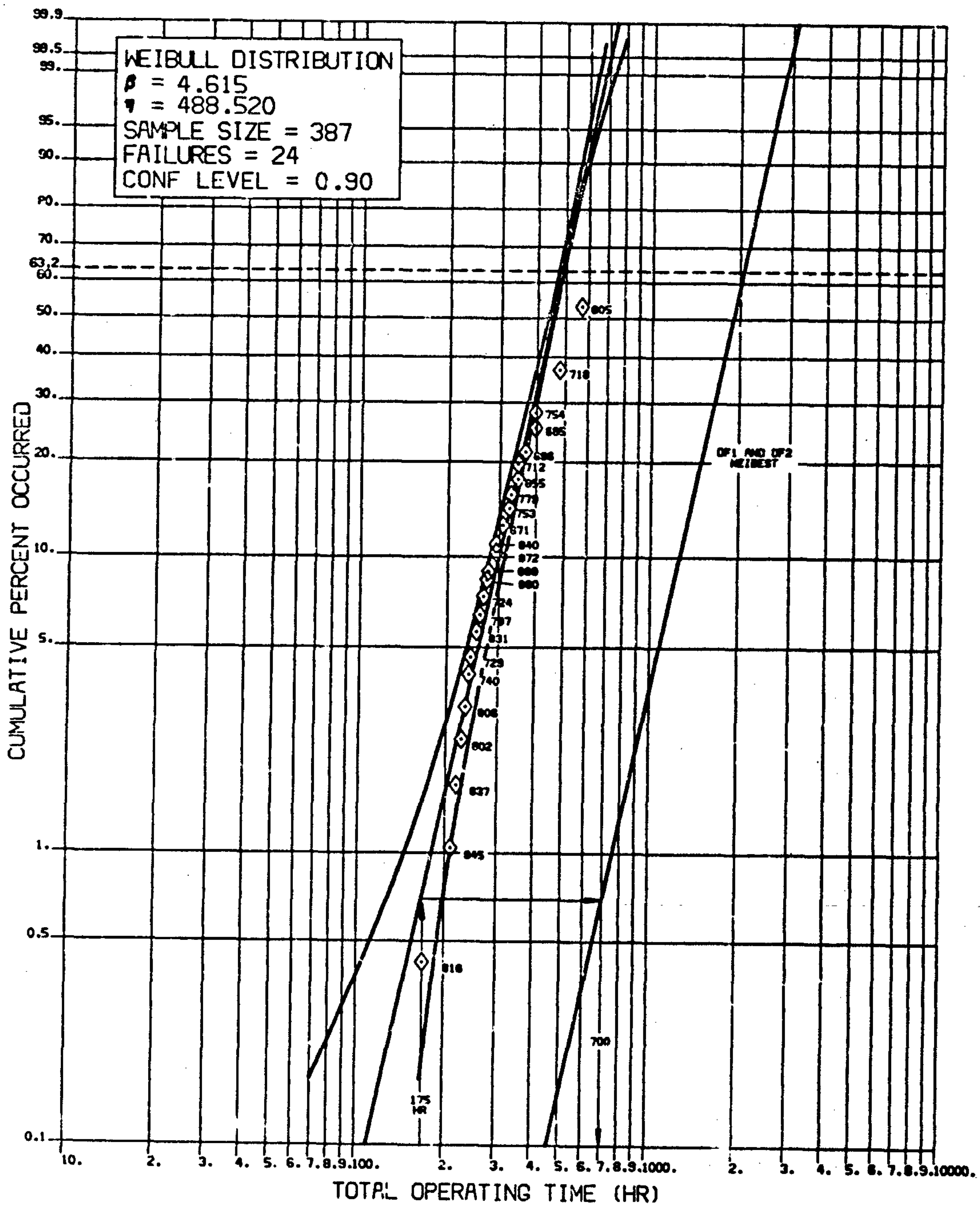


Figure 4.6. When Should We Pull the Suspect Batch?

CHAPTER 5

SUBSTANTIATION AND RELIABILITY TESTING

5.1 FOREWORD

The objective of this chapter is to address the statistical requirements of substantiation and reliability testing when the underlying failure distribution is Weibull. Substantiation testing demonstrates that a redesigned part or system has eliminated or significantly improved a known failure mode (β and η are assumed to be known). Reliability testing demonstrates that a reliability requirement has been met.

It is assumed in the reliability testing section that the Weibull slope parameter, β, is known. If the failure distribution is known to be Weibull but β is unknown, test plans developed by K. Fertig and N. Mann[1] may be used.

A test plan gives the required number of units and the amount of time to be accumulated on each to either substantiate a fix or meet a reliability goal. It also gives a success criterion, where the test is passed if the success criterion is met. In a zero-failure test plan the success criterion is no failure: the test is passed if every unit runs the prescribed amount of time and no unit fails while on test.

Test plans can also be generated with a 1-failure success criterion, a two-failure success criterion, etc. But all of these plans require more testing than the zero-failure plan.

A measure of confidence is usually built into statistically designed test plans, guaranteeing that if the failure mode in question has not been fixed or the reliability requirement has not been achieved, there is a low probability that the test will be passed. The zero-failure test plans in this chapter guarantee with 90% confidence that the test will be failed if the required goal has not been achieved. Thus, a part or system will have at most a 10% chance of being accepted as satisfactory when in fact it is not.

5.2 ZERO-FAILURE TEST PLANS FOR SUBSTANTIATION TESTING

A ball and roller bearing system has a Weibull failure mode, unbalance, with β (the Weibull slope parameter) equal to 2, and η equal to 500 hours. The system is redesigned, and three redesigned systems are available for testing. How many hours should each system be tested to demonstrate that this mode of unbalance has been eliminated or significantly improved?

The Weibull plot in Figure 5.1 illustrates the time-to-unbalance distribution.

Table 5.1 is used to answer this type of question. It is entered with the value of β and the number of units to be tested. The corresponding table entry is multiplied by the characteristic life to be demonstrated to find the test time required of each unit.

[1] Mann, N. R. and K. W. Fertig, (1980) Life-Test Sampling Plans for Two-Parameter Weibull Populations. Technometrics, 22, 165-177.

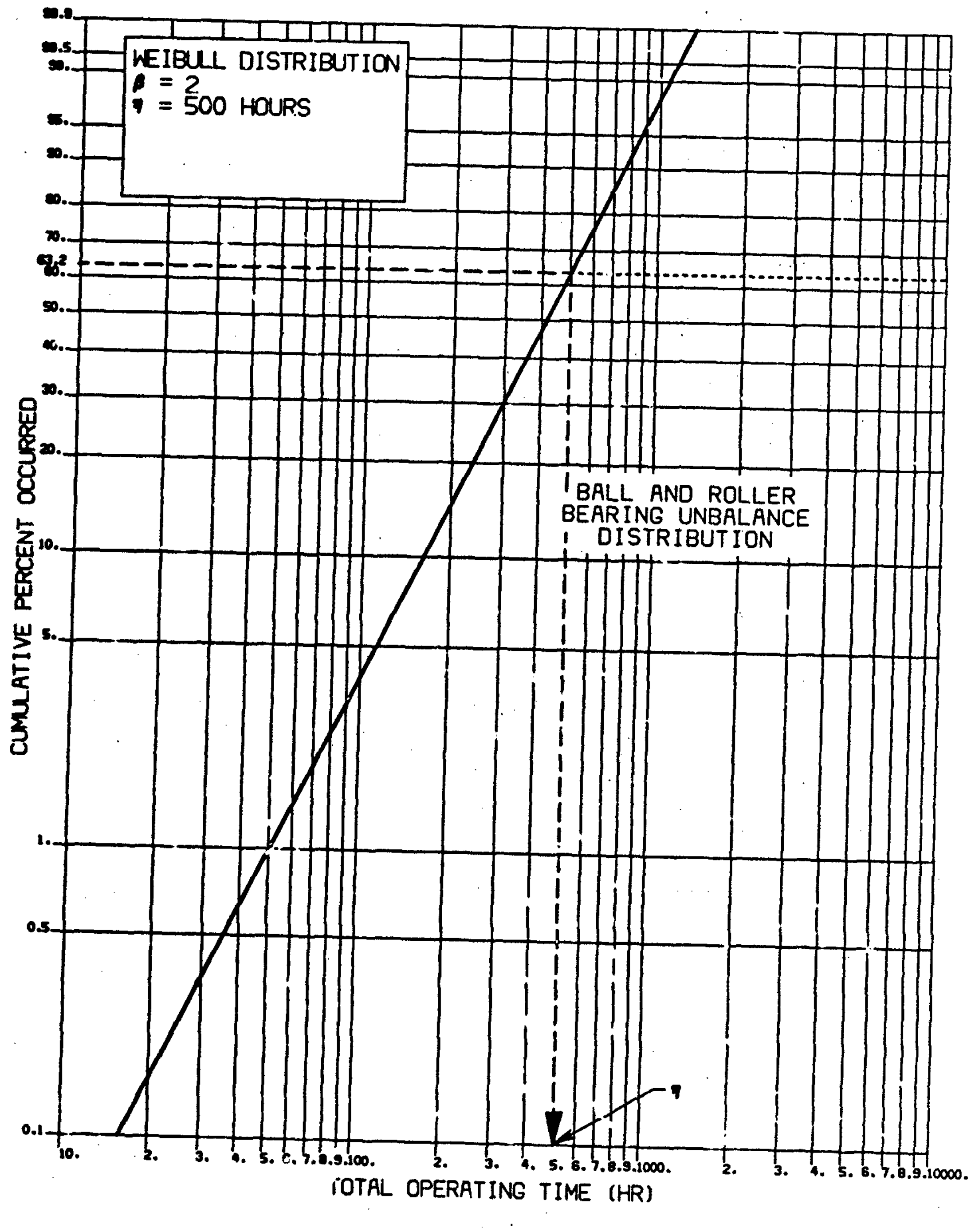


Figure 5.1. Ball and Roller Bearing Unbalance Distribution

TABLE 5.1 CHARACTERISTIC LIFE MULTIPLIERS FOR ZERO-FAILURE TEST PLANS
CONFIDENCE LEVEL: 0.90

Sample Size	*β* 0.5 *Infant Mortality*	1.0 *Random*	1.5 +	2.0 ------	2.5 *Gradual Wearout*	3.0 ------	3.5 +	4.0 *Rapid Wearout (Brick Wall)* +	4.5 –	5.0 – +
3	0.589	0.767	0.838	0.876	0.900	0.916	0.927	0.936	0.943	0.948
4	0.331	0.576	0.692	0.759	0.802	0.832	0.854	0.871	0.884	0.895
5	0.212	0.460	0.596	0.679	0.733	0.772	0.801	0.824	0.842	0.856
6	0.147	0.384	0.528	0.619	0.682	0.727	0.761	0.787	0.808	0.826
7	0.108	0.329	0.477	0.574	0.641	0.690	0.728	0.757	0.781	0.801
8	0.083	0.288	0.436	0.536	0.608	0.660	0.701	0.732	0.758	0.780
9	0.065	0.256	0.403	0.506	0.580	0.635	0.677	0.711	0.739	0.761
10	0.053	0.230	0.376	0.480	0.556	0.613	0.657	0.693	0.722	0.745
12	0.037	0.192	0.333	0.438	0.517	0.577	0.624	0.662	0.693	0.719
14	0.027	0.164	0.300	0.406	0.486	0.548	0.597	0.637	0.670	0.697
16	0.021	0.144	0.275	0.379	0.461	0.524	0.575	0.616	0.650	0.679
18	0.016	0.128	0.254	0.358	0.439	0.504	0.556	0.598	0.633	0.663
20	0.013	0.115	0.237	0.339	0.421	0.486	0.539	0.582	0.619	0.649
25	0.008	0.092	0.204	0.303	0.385	0.452	0.506	0.551	0.589	0.621
30	0.006	0.077	0.181	0.277	0.358	0.425	0.480	0.526	0.565	0.598
40	0.003	0.058	0.149	0.240	0.319	0.386	0.442	0.490	0.530	0.565
50	0.002	0.046	0.128	0.215	0.292	0.358	0.415	0.463	0.505	0.540

In the ball and roller bearing example, Table 5.1 is entered with β equal to 2.0 and a sample size of three. The corresponding table entry is 0.876. The characteristic life to be demonstrated is 500 hours. The number of hours that each system should be tested is:

$$0.876 \times 500 \text{ hours} = 438 \text{ hours.}$$

Thus, the zero-failure test plan to substantiate the ball and roller bearing system fix is: test three systems for 438 hours each. If all three systems are in balance at the end of the test, then the unbalance mode was either eliminated or significantly improved (with 90% confidence).

If there is a constraint on the amount of test time accumulated on each unit, Table 5.2 is used to determine the number of units required for the test. For example, suppose in the previous example that no more than 300 hours could be accumulated on any bearing system. Table 5.2 is entered with the known value of β and the ratio of the test time to the characteristic life being substantiated. In the ball and roller bearing system example, Table 5.2 is entered with β equal to 2.0 and the ratio

$$\frac{\text{300 test hours per system}}{\text{500 hour characteristic life}} = 0.6$$

The corresponding entry in Table 5.2 is seven. The resulting test plan is: test seven systems for 300 hours each. If all seven systems are in balance at the end of the test, then the unbalance mode was either eliminated or significantly improved (with 90% confidence).

TABLE 5.2 REQUIRED SAMPLE SIZES FOR ZERO-FAILURE TEST PLANS
CONFIDENCE LEVEL: 0.90

	β									
	0.5				*2.5*			*4.0*	*4.5*	*5.0*
	Infant	*1.0*	*1.5*	*2.0*	*Gradual*	*3.0*	*3.5*	*Rapid Wearout*		
Ratio	*Mortality*	*Random*	+	------	*Wearout*	------	+	+ -	*(Brick Wall)*	- +
0.01	24	231	2303	23025	*****	*****	*****	*****	*****	*****
0.02	17	116	815	5757	40703	*****	*****	*****	*****	*****
0.03	14	77	444	2559	14771	85278	*****	*****	*****	*****
0.04	12	58	288	1440	7196	35977	*****	*****	*****	*****
0.05	11	47	206	922	4119	18420	82377	*****	*****	*****
0.06	10	39	157	640	2612	10660	43519	*****	*****	*****
0.07	9	33	125	470	1777	6713	25373	95898	*****	*****
0.08	9	29	102	360	1272	4498	15900	56214	*****	*****
0.09	8	26	86	285	948	3159	35094	35094	*****	*****
0.10	8	24	73	231	729	2303	7282	23025	72812	*****
0.20	6	12	26	58	129	288	644	1440	3218	7196
0.30	5	8	15	26	47	86	156	285	519	948
0.40	4	6	10	15	23	36	57	90	143	225
0.50	4	5	7	10	14	19	27	37	53	74
0.60	3	4	5	7	9	11	14	18	23	30
0.70	3	4	4	5	6	7	9	10	12	14
0.80	3	3	4	4	5	5	6	6	7	8
0.90	3	3	3	3	3	4	4	4	4	4
1.00	3	3	3	3	3	3	3	3	3	3

***** Indicates sample size exceeds 100,000

5.3 ZERO-FAILURE TEST PLANS FOR RELIABILITY TESTING

This section contains zero-failure test plans for demonstrating a reliability goal when the underlying failure distribution is Weibull with known slope parameter β. A turbine engine combustor's reliability goal was 99% reliability at 1800 cycles under service-like conditions. Success was defined as a combustor having no circumferential cracks longer than 20 inches (out of a possible 53 inches). The number of cycles required to reach a 20-inch crack was known to follow a Weibull distribution with β equal to 3. How many combustors must be tested, and how many cycles must each accumulate, to demonstrate this goal with a high level of confidence?

First, the reliability goal is re-expressed as a characteristic life goal, and then the test plan is designed.

Re-expression of Reliability Goal

Reliability requirements generally assume one of the following forms:

Form 1: The reliability of the unit is required to be at least X% after a certain number of hours or cycles of life. (This is equivalent to the percent failing being at most 100-X%). The Weibull plot in Figure 5.2 illustrates the requirement of at least 99% reliability (at most 1% unreliability) at 1000 hours.

Form 2: The B10 life (or B1 life, or B.1 life, etc.) is required to be at least X hours or cycles. By definition, the unit has a 10% chance of failing before reaching its B10 life, a 1% chance of failing before reaching its B1 life, etc. See Figure 5.3.

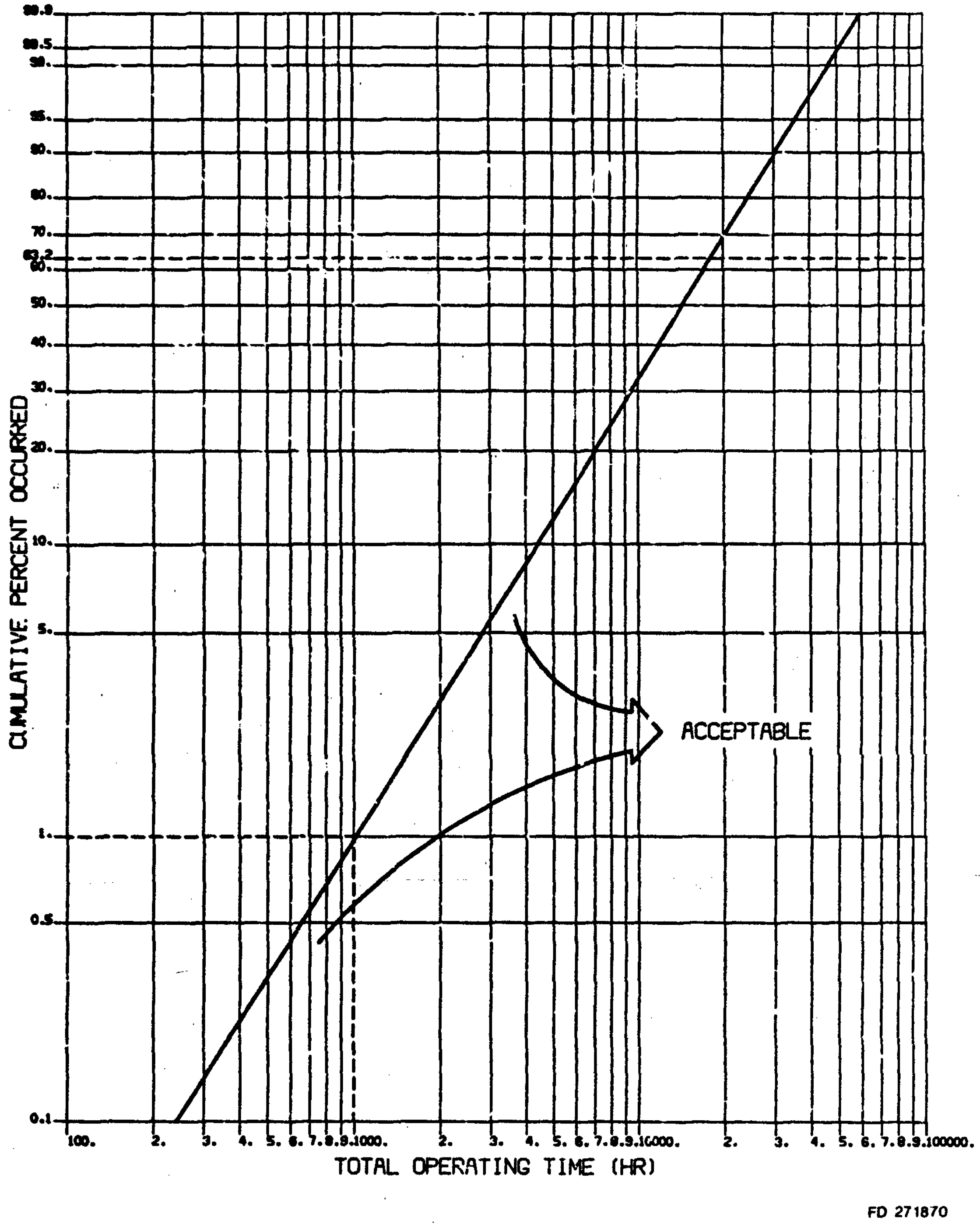


Figure 5.2. Weibull Plot for 0.99 Reliability at 1000 hr

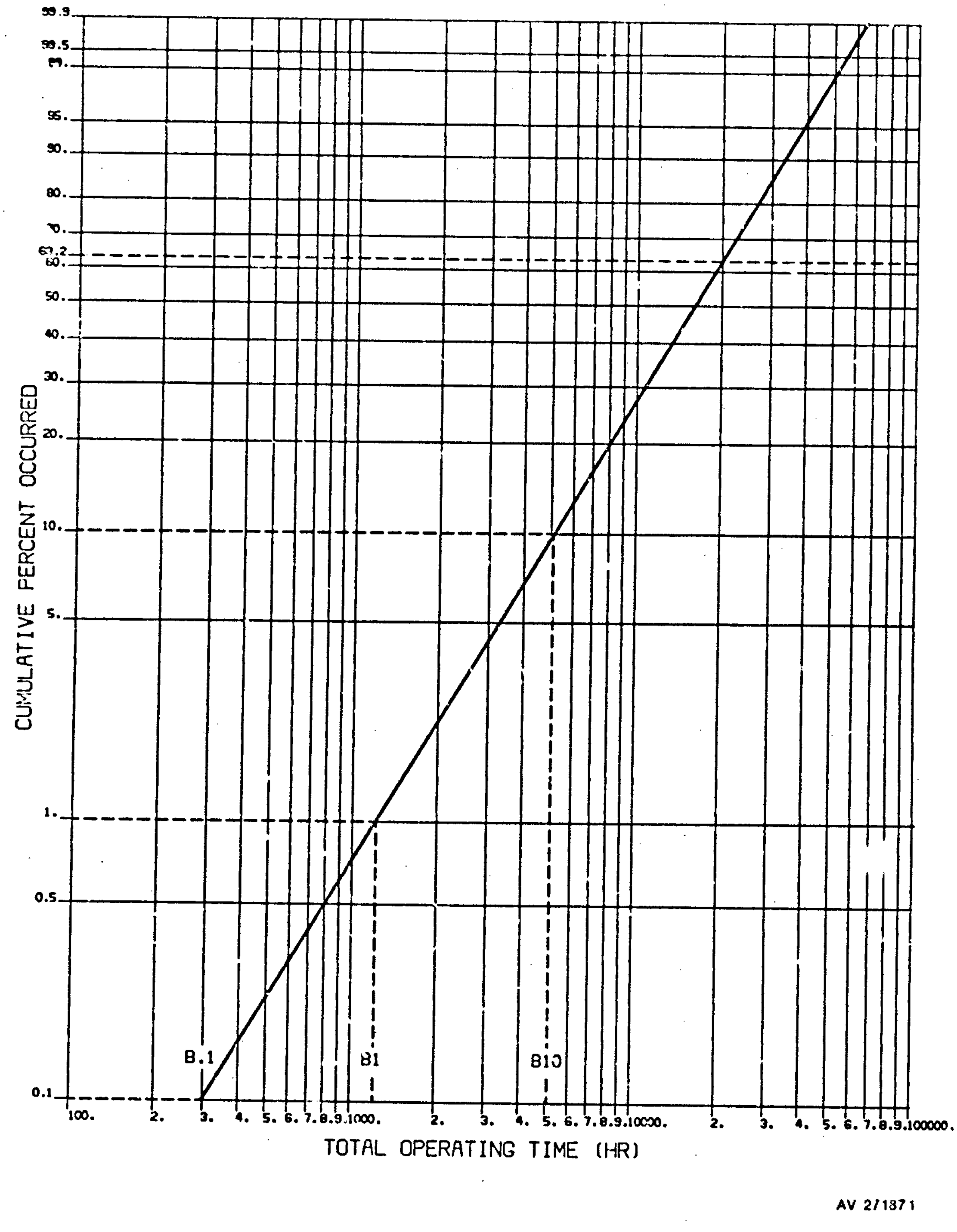


Figure 5.3. Illustration of B.1, B1, and B10 Lives

Figure 5.4 illustrates the requirement of a 7000 hour B10 life.

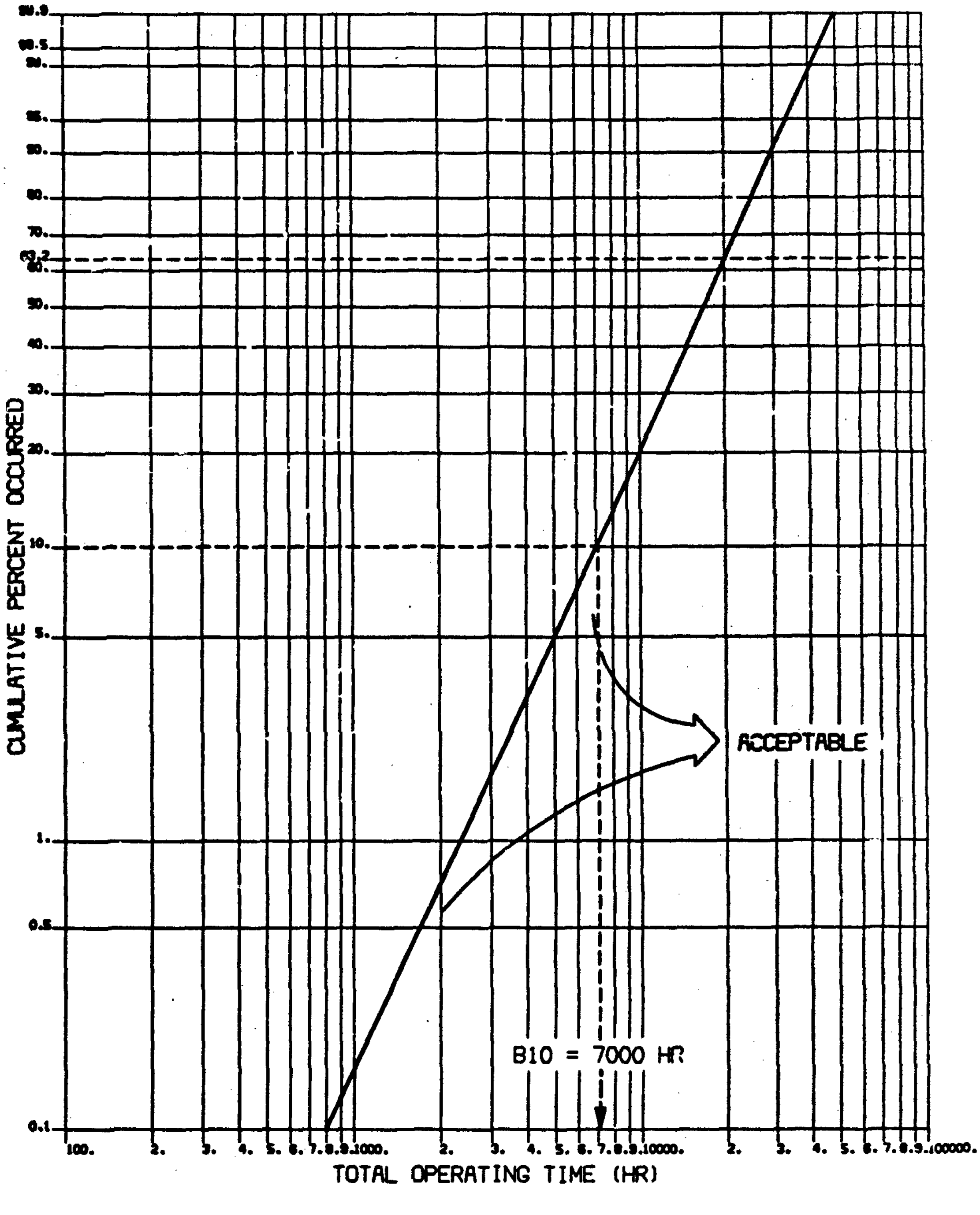


Figure 5.4. Illustration of 7000 hr B10 Life

Reliability requirements assuming either of these two forms can be expressed as a minimum characteristic life requirement. Given that the time-to-failure distribution is Weibull, with a known β, reliability at time t is a function of η:

$$R(t) = e^{-(t/\eta)^\beta} \tag{5.1}$$

This expression can be rearranged algebraically, giving

$$\eta = \frac{t}{|-\ln R(t)|^{1/\beta}} \tag{5.2}$$

Equation (5.2) can be used to express either form of reliability requirement in terms of η. If the requirement is, for example, that the reliability of the turbine engine combustor must be at least 0.99 at 1800 cycles ($\beta = 3$), then substituting t = 1800 and R(t) = 0.99 into equation (5.2) gives

$$\eta = \frac{1800}{|-\ln(0.99)|^{1/3}}$$

or $\eta = 8340.9$

The 0.99 reliability requirement is equivalent to the requirement that η be at least 8340.9 cycles. See Figure 5.5.

Similarly, if the requirement is a B10 life of 2000 hours, then substituting t = 2000 and R(t) = 0.90 into equation (5.2) gives

$$\eta = \frac{2000}{|-\ln(0.90)|^{1/2}}, \text{ assuming } \beta = 2$$

or $\eta = 6161.6$

Thus, the B10 life requirement of 2000 hours with $\beta = 2$ is equivalent to the requirement that η is at least 6161.6 hours. See Figure 5.6.

Designing Test Plans

Once the minimum characteristic life requirement has been calculated, Tables 5.1 and 5.2 can be used to design the test plan.

In the combustor reliability example, the 99% reliability goal at 1800 cycles was re-expressed as an 8340.9 cycle characteristic life goal. Ten combustors were available for this reliability demonstration test. To find the test cycles required of each combustor, enter Table 5.1 with β equal to 3.0 and a sample size of 10. The corresponding table entry is 0.613. Multiply the table entry by the characteristic life requirement to find the test time required of each unit.

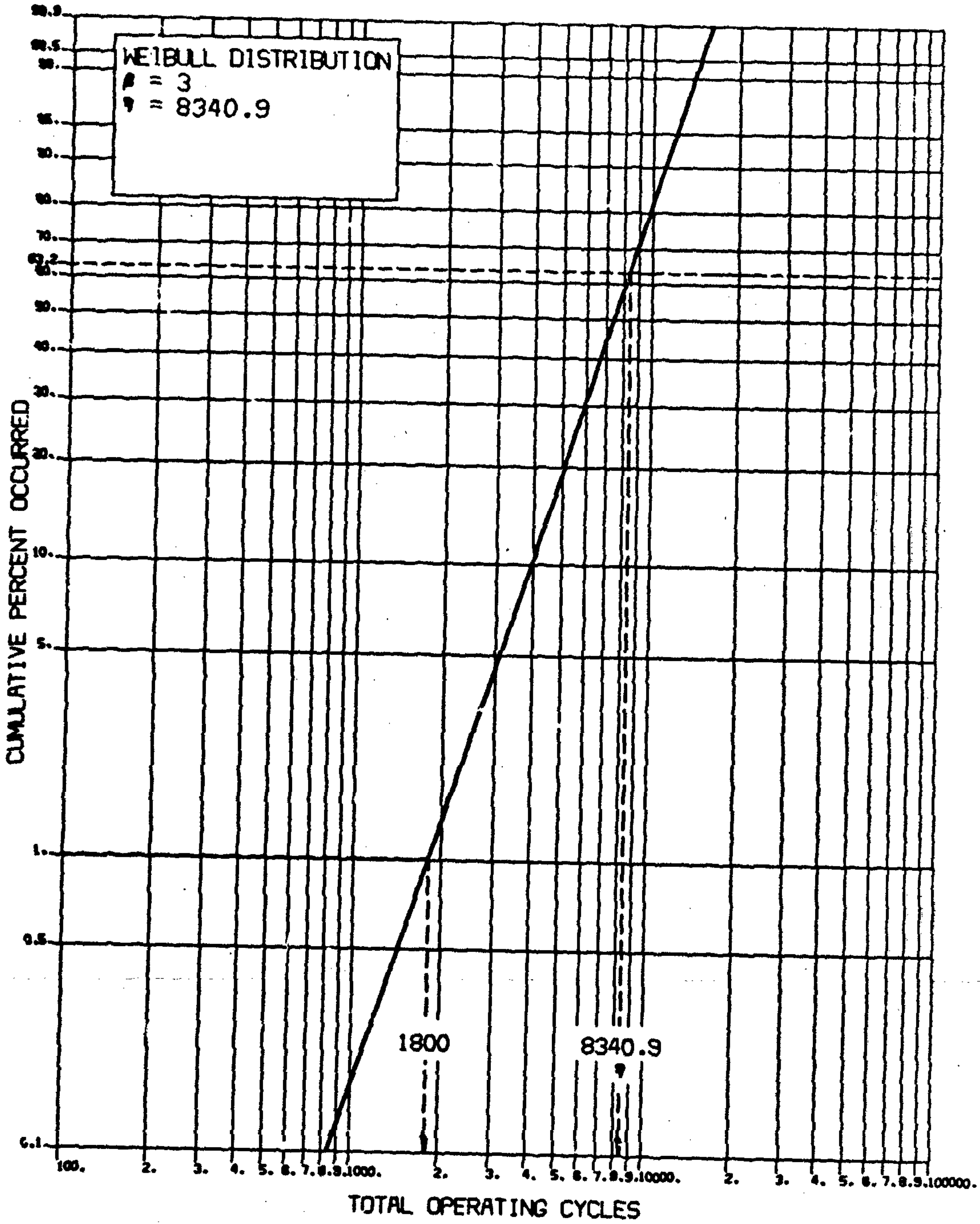


Figure 5.5. A Reliability of 0.99 at 1800 Cycles Is Equivalent to an 8340.9 Cycle Characteristic Life When β=3

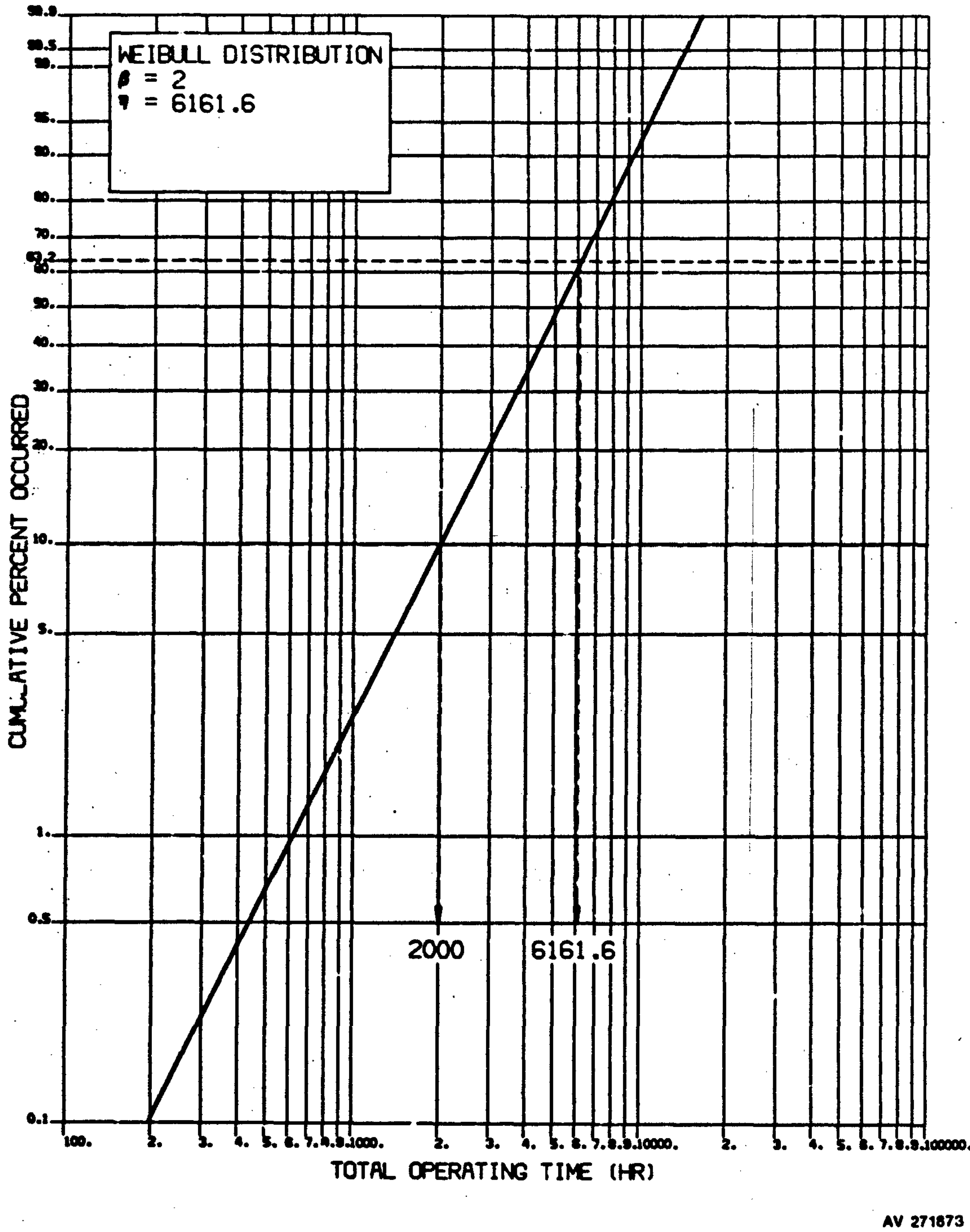


Figure 5.6. A 2000 hr B10 Life Is Equivalent to a 6161.6 hr Characteristic Life When $\beta = 2$

In the combustor example, multiplying the Table 5.1 entry of 0.613 by the characteristic life requirement of 8340.9 cycles gives a test time of:

$$0.613 \times 8340.9 \text{ cycles} = 5113.0 \text{ cycles}.$$

Thus, the zero-failure test plan to demonstrate 99% reliability at 1800 cycles requires testing 10 combustors for 5113 cycles each. If no combustor develops a circumferential crack longer than 20 inches, then the test is passed.

How many combustors are required if each can accumulate at most 750 test cycles? To answer this, enter Table 5.2 with the assumed value of β, the Weibull slope parameter, and the ratio of the test time to the calculated characteristic life requirement. In the combustor example, β was assumed to be 3.0, and the ratio of the test time to the calculated characteristic life requirement is:

$$\frac{750 \text{ test cycles per combustor}}{8340.9 \text{ cycles characteristic life}} = 0.09$$

The corresponding entry in Table 5.2 is 3159. The resulting test plan requires testing 3159 combustors for 750 cycles each. If no combustor develops a circumferential crack longer than 20 inches, then the test is passed.

5.4 TOTAL TEST TIME

Two reliability test plans were constructed in Section 5.3 to demonstrate that a characteristic life was at least 8340.9 cycles, with 90% confidence.

	Number of Combustors	*Test Cycles Per Combustor*	*Total Test Cycles*
Plan 1	3,159	750	3,159 × 750 = 2,369,250
Plan 2	10	5,113	10 × 5,113 = 51,130

Note that, in terms of total test cycles, it is more efficient to run the smaller number of combustors for a greater number of cycles. Plan 2 demonstrates the same reliability as Plan 1, but requires fewer total test cycles.

This efficiency is realized for every test plan in this section where β exceeds 1.0.

The situation is reversed for β less than 1. In this case, the greater the number of units on test, the lower is the total test time.

When β is 1, the total test time is constant, regardless of the number of items on test.

5.5 ADVANTAGES AND LIMITATIONS OF THE ZERO-FAILURE TEST PLANS

The test plans introduced in Section 5.2 limit the probability that substandard reliability units will pass the tests. This is generally the most important goal in reliability testing. Also, the test plans are simple and easy to use.

However, they are only designed to limit the acceptance of substandard reliability items. They do not control the probability that units from a high reliability design will pass the tests.

In some instances, the experimenter is interested in guaranteeing a high probability of acceptance for high reliability units.

For example, to demonstrate that a characteristic life is at least 2000 hours with 90% confidence, assuming β = 2.5, requires that 14 units be tested 1000 hours without failure. (Enter Table 5.2 with β = 2.5 and the ratio 1000 test hours per unit/2000 hours = 0.5 to find 14 units.) Another requirement might be that designs with characteristic lives greater than 4000 hours pass the test with at least 90% probability. The zero-failure test gives these high-reliability designs only a 65% chance of passing. (See Figure 5.7.)

There are two remedies for this problem. The minimum characteristic life requirement can be reduced enough to guarantee a suitably high probability of acceptance for high reliability units, or the size of the test can be increased until both requirements are met. The second option is considered in Sections 5.6 through 5.8.

The curves in Figure 5.7 assist the experimenter in determining how much to reduce the minimum characteristic life requirements to meet high reliability requirements. They give the probability of passing the zero-failure test as a function of the Weibull parameter β and the ratio of the characteristic life of interest to the minimum required characteristic life. For example, the probability of successfully completing the zero-failure test mentioned earlier in this section for units whose characteristic life is 4000 hours is 0.65. To see this, enter the β = 2.5 curve of Figure 5.7 with the x-axis ratio of:

$$R = \frac{4000 \text{ hours (characteristic life of interest)}}{2000 \text{ hours (minimum characteristic life requirement)}}$$

or R = 2.0.

The probability of successfully completing the test, from Figure 5.7, is 0.65.

In the preceding example, suppose the characteristic life requirement were dropped from 2000 hours to 1250 hours. Only five units would have to be tested 1000 hours, instead of 14. (Enter Table 5.2 with β = 2.5 and the ratio (1000 test hours per unit/1250 hour characteristic life requirement) = 0.8, to get the five-unit requirement). To find the probability of acceptance of the 4000-hour characteristic life design, enter the β = 2.5 curve in Figure 5.7 with a ratio of 4000 hours/1250 hours = 3.2. The chances of passing are 88% — close to the 90% requirement.

Reliability demonstration tests that terminate successfully with no failure have one other advantage. Very high reliability often makes a demonstration test-to-failure impractical. In this case, a zero-failure test plan is desirable. The risk is that unless some units are run to failure, the statistical assumptions inherent in the test design cannot be validated. (For example, with failed units, the Weibull slope parameter β can be estimated and compared to the assumed value of β.)

5.6 NON-ZERO-FAILURE TEST PLANS

In Section 5.2, test plans were introduced to demonstrate that a lower limit characteristic life had been achieved, with 90% confidence. The plans assume that the unit's time-to-failure distribution is Weibull with known slope parameter β.

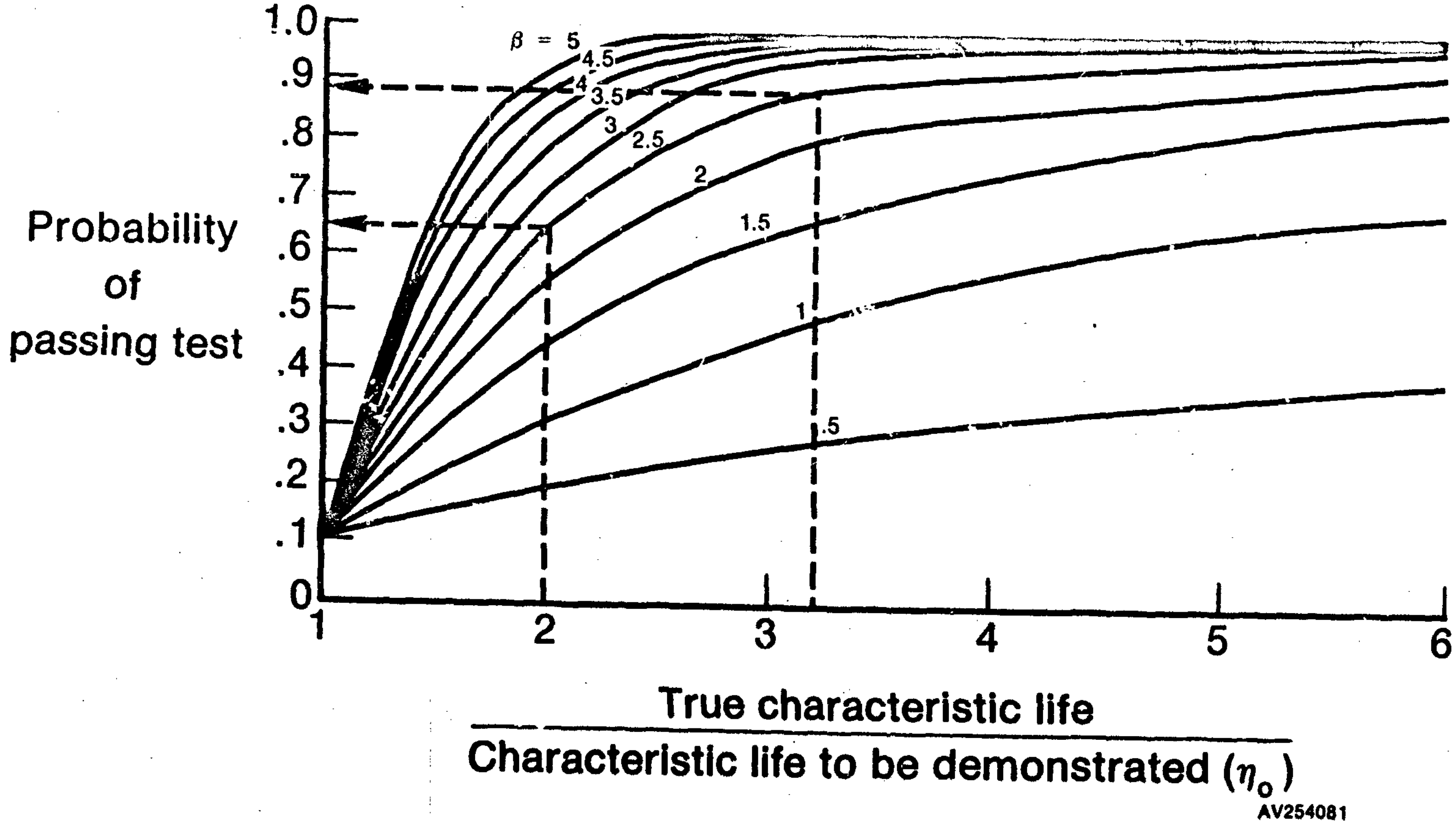


Figure 5.7. Probability of Passing the Zero-Failure Tests for β's Between 0.5 and 5

As discussed in Section 5.5, these tests do not control the risk of rejecting units with acceptably high characteristic lives (called producer's risk). They only control the risks of passing the test with a characteristic life below the lower limit (called consumer's risk). This risk was set at 10%.

The methods for test plan construction introduced in this section provide control over both forms of risk if the zero-failure plans do not adequately balance the two. The methods can be found in "*Methods for Statistical Analysis of Reliability and Life Data*"[1].

5.7 DESIGNING TEST PLANS

These tests will have the following structure:

A. Put n items on test for t hours (cycles) each.
B. When an item on test fails, it is not replaced.
C. If r_0 or fewer failures occur, the test is passed.

This section describes methods for calculating r_0 and n satisfying the two constraints:

A. The probability of passing the test with a characteristic life as low as η_0 should be no more than α_0 (minimum life requirement).

B. The probability of passing the test with a characteristic life as high as η_1 should be at least α_1.

η_0 is the characteristic life to be demonstrated. η_1 is sometimes referred to as the "design" characteristic life. α_0 is usually set at 0.05 or 0.1, and α_1 is usually set at 0.9 or 0.95.

The equations to be introduced require the definition of some standard mathematical notation.

A. Summation

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + - - - + x_n$$

[1] Mann, Nancy R., Ray E. Schafer, and Nozer D. Singpurwalla (1974), Methods for Statistical Analysis of Reliability and Life Data, John Wiley and Sons, New York, Chapter 6, pp 312-315 and p 328.

B. Factorial

$$n! = n\cdot(n-1)\cdot(n-2)\cdots\cdots\cdot 2\cdot 1$$

C. Number of subsets of size r from a set of n

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Assuming that the time-to-failure distribution of the items on test is Weibull, with known parameter β, the following equations should be solved for r_0 and n to satisfy the two "probability of passing" requirements.

$$\alpha_0 = \sum_{r=0}^{r_0} \binom{n}{r} p_0^r (1-p_0)^{n-r} \qquad (5.3)$$

$$\alpha_1 = \sum_{r=0}^{r_0} \binom{n}{r} p_1^r (1-p_1)^{n-r} \qquad (5.4)$$

where $p_0 = 1-e^{-(t/\eta_0)^\beta}$

$p_1 = 1-e^{-(t/\eta_1)^\beta}$

t is the test time per unit

n is the number of units on test

r_0 is the allowable number of failures

η_0 is the demonstrated characteristic life

η_1 is the design characteristic life

β is the assumed value of the Weibull slope parameter

α_0 is the probability of passing the test with a characteristic life equal to η_0 (α_0 is set by the experimenter)

α_1 is the probability of passing the test with a characteristic life equal to η_1 (α_1 is set by the experimenter)

Equations (5.3) and (5.4) generally require a computer for their solution. Certain computer packages are available that solve these equations. Dr. K. E. Case of the Oklahoma State University School of Industrial Engineering and Management (Stillwater, Oklahoma) built an interactive program that includes the ability to solve equations (5.3) and (5.4.)[1]

Equations (5.3) and (5.4) generally cannot be solved for a combination of n and r_0 that satisfy the target probabilities α_0 and α_1 exactly. Some authors recommend solving the equations so that the actual probability α_0' of passing the test with $\eta = \eta_0$ is no greater than α_0, and the corresponding true probability α_1' is at least as great as α_1. The next section discusses the method recommended by Dr. Case[1] for solving equations (5.3) and (5.4).

[1] Case, Kenneth E. and Lynn L. Jones (1979), "An Interactive Computer Program for the Study of Attributes Acceptance Sampling. Final Technical Report," Oklahoma State University, School of Industrial Engineering and Management, Stillwater, Oklahoma.

5.8 RECOMMENDED METHOD FOR SOLVING EQUATIONS

Dr. Case's final technical report[1] describes the recommended method. It consists of the following steps:

1. Calculate $p_0 = 1 - e^{-(t/\eta_0)^\beta}$
   $p_1 = 1 - e^{-(t/\eta_1)^\beta}$

2. Set $r_0 = 0$

3. Find the values of n that satisfy equations 5.3 and 5.4. Call them n_0 and n_1, respectively.

4. Calculate $a = \frac{n_0 p_0}{n_1 p_1}$ and $b = p_0/p_1$.

5. If a is greater than b, increase r_0 by 1, and repeat steps 3, 4, and 5.

6. Continue the process until two contiguous values of r_0 are found whose calculated a-ratios bound b.

7. Select as the final value of r_0 that which has the a-ratio nearer the desired ratio $b = p_0/p_1$.

8. For the selected value of r_0, there are two values of n, n_0 and n_1, calculated in step 3. Average n_0 and n_1 to get the final sample size:

$$n = \frac{n_0 + n_1}{2}$$

5.9 PROBLEMS

Problem 5-1

A turbine engine exhaust nozzle control bearing was failing prematurely due to fatigue. Bearing failures followed a Weibull distribution with β equal to 1.5 (a common value for bearing fatigue) and η equal to 3000 hours. The bearing was redesigned, and the environment in which it operated was improved to give the bearing a higher expected life. Twenty redesigned bearings were available for testing. How long should each be tested to demonstrate, with 90% confidence, that the fatigue mode was significantly improved?

Problem 5-2

High pressure turbine vanes were eroding beyond allowable limits. A significant percentage of the engines in service were being removed for vane repair or replacement prior to their scheduled turbine maintenance. The time to failure — determined by the worst vane in the set — followed a Weibull distribution with $\beta = 3$ and $\eta = 1300$ cycles.

Through redesign and material changes the vane's durability was improved. Design a test to demonstrate the new vane's goal: no more than 5% of the engines should be removed by 2300 cycles for vane erosion (with 90% confidence). During this test, assume that the turbines are

[1] Case, Kenneth E. and Lynn L. Jones (1979), "An Interactive Computer Program for the Study of Attributes Acceptance Sampling. Final Technical Report," Oklahoma State University, School of Industrial Engineering and Management, Stillwater, Oklahoma.

limited to running at most 5000 cycles each. Also, assume that the time to engine removal for excessive vane erosion would still follow a Weibull distribution with $\beta = 3$.

Problem 5-3:

In Section 5.5, the zero-failure test plan was given to demonstrate that the characteristic life of a Weibull distribution with $\beta = 2.5$ is at least 2000 hours, with 90% confidence. It required that 14 units be tested 1000 hours. The test is passed if none of the 14 units fails during the 1000 hours of testing.

The additional requirement was added that units with characteristic lives greater than 4000 hours should pass the test with at least 90% probability. It was shown that the zero-failure test plan could only guarantee a 65% chance of passing.

Use the methods introduced in Section 5.8 to construct a test satisfying all of the above requirements.

Solutions to these problems are in Appendix J.

CHAPTER 6

CASE HISTORIES WITH WEIBULL APPLICATIONS

6.1 FOREWORD

This chapter provides examples of Weibull analysis used in a variety of situations. The examples were chosen from studies which include the complete cycle of analysis, deduction, recommendation, and implementation. The case studies selected are:

(1) Turbopump Bearing Failures
(2) Gearbox Housing Cracks
(3) Opportunistic Maintenance Screening Intervals
(4) Support Cost Model
(5) Vane and Case Cracking.

6.2 EXAMPLE 1: TURBOPUMP BEARING FAILURES

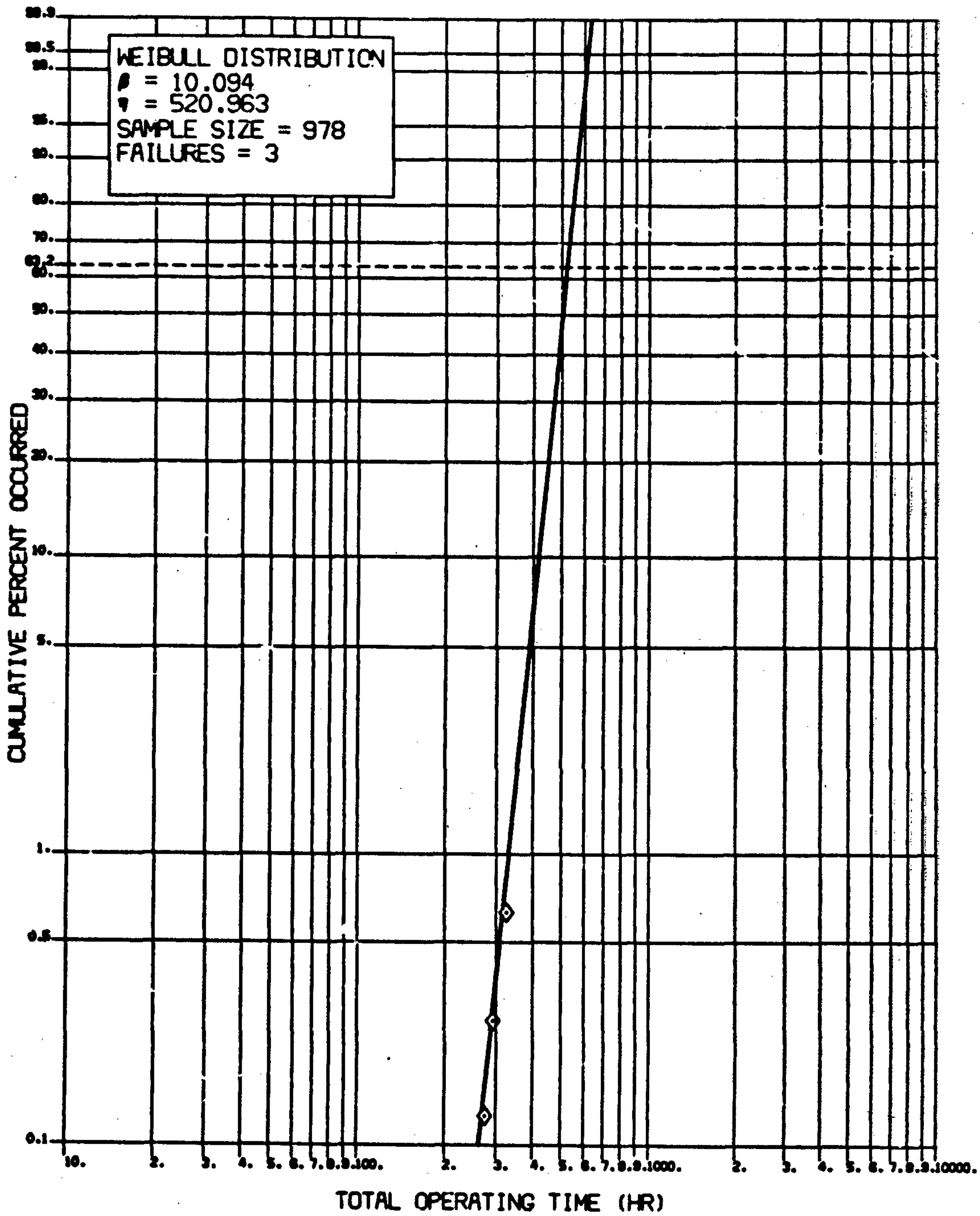
When this study began, three failures of the augmentor turbopump of an aircraft fighter engine had occurred in the field. This was an urgent problem because the failure enabled fuel to escape and ignite. Because of this hazard, top priority was assigned to the analysis of data that might help resolve this problem.

6.3 INITIAL ANALYSIS — SMALL SAMPLE

The first analysis was the evaluation of the three failures through Weibull analysis. Note that this was an extremely small sample from the 978 turbopumps that were operating in the fleet. The data were ranked by turbopump operating time, treating the successful pumps as censored units. The resulting Weibull plot is shown in Figure 6.1.

Even with this small sample some valuable observations could be made. First, the very steep slope, $\beta = 10$, indicates that the failure mode is one of rapid wearout preceded by a relatively safe period. Inspection of Figure 6.1 shows that the probability of a turbopump failure prior to 200 hours is negligible, but after 250 hours the probability increases rapidly.

A second inference can be made from the initial Weibull analysis. The very steep slope (β 10) along with the existence of many unfailed pumps with run times greater than the failed pumps suggests that the failed pumps are part of a unique batch. The method used to determine whether or not a given failure mode is a batch problem is to evaluate the Weibull equation with the parameters calculated (Figure 6.1) for each successful and failed turbopump. For each pump, the probability of failure is determined from the Weibull equation and these probabilities are then summed. If the failure mode applies to the entire fleet, the sum of the cumulative probabilities should approximate the number of failures observed, in this case 3. For example:



FD 271875

Figure 6.1. Weibull Plot for Augmentor Pump Bearing

$$\Sigma F(t_i) = \Sigma[1 - e^{(t_i/\eta)^{\beta}}] \tag{6.1}$$

where:

$\Sigma F(t_i)$ = sum of probabilities of each unit
t_i = time on each unit (both failed and unfailed)
η = 520.963 = characteristic life
β = 10.094 = slope of Weibull
e = exponential (base of natural logarithms).

However, with these data the answer was 117 failures, indicating that the failure mode applied to less than the entire fleet of turbopumps. Recommendations were made to Project Engineering that the turbopump vendor and the bearing vendor should review their processes to determine if anything had changed, either in the process, the material, or the assembly. Initially, no change was found that supported the batch hypothesis.

6.4 TWO MONTHS LATER — BATCH IDENTIFIED

At this point in the analysis there were seven confirmed and two unconfirmed failures. It was observed that the serial numbers of the failed pumps were all quite high, ranging from No. 671 to No. 872 in the sample of approximately 1000 pumps. The closeness of serial numbers supported the hypothesis that this was a batch problem. If it is assumed that the batch started at the first failed part, Serial No. 671, and extended to the latest pumps produced, the Weibull equation generated fewer than nine failures. By iterating, it was found that by starting at Serial No. 650 nine failures were generated, corresponding to the seven observed and two unconfirmed failures. (See Figure 6.2.) This indicated there were about 353 pumps in the batch.

6.5 RISK PREDICTION

With a serious problem involving approximately 350 pumps, the next step was to forecast the number of failures which could be expected in the near future. The risk analysis was performed using the methods described in Chapter 3, and was limited to 353 suspect pumps.

The total operating time on engines is kept in a data system that is updated monthly. It is also known that each pump accumulates an average of 25 hours operating time per month. The risk analysis is illustrated in Figure 6.3. With the 353 pump times for the Weibull curve in Figure 6.2, a cumulative total of 9.17 failures can be calculated for the "now" time using the method explained in Chapter 3. Increasing each pump's time by 25 hours and again accumulating the probabilities of failure, the value of 12.26 was obtained. The delta between 9.17 and 12.26 indicated that approximately three more failures were expected in the next month. This analysis covered 24 months of operation and the results are presented in Table 6.1.

As the forecast indicates, almost all of the suspect lot was expected to fail within a little more than two years. This was obviously a serious problem if the analysis was correct.

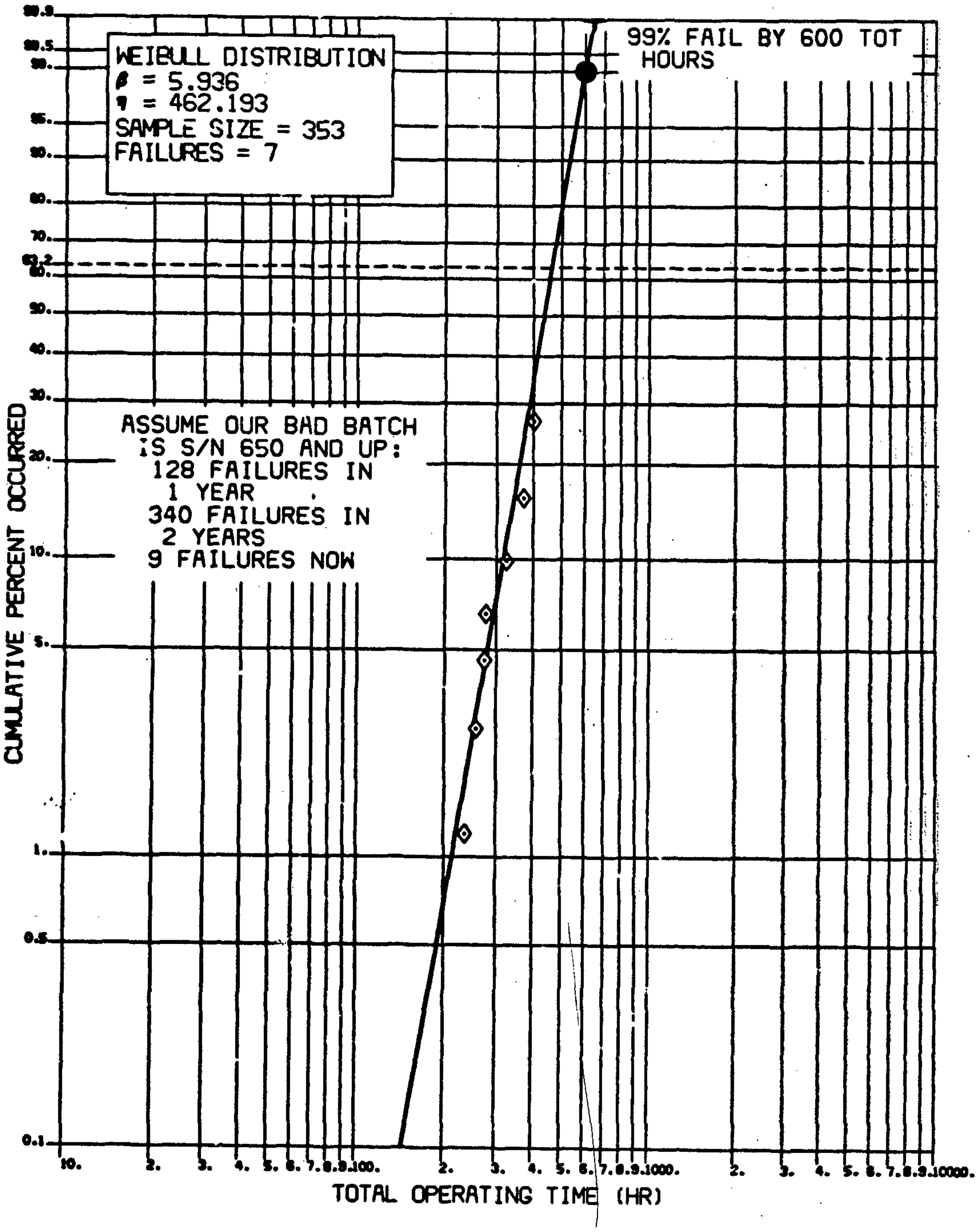


Figure 6.2. Augmentor 650 on Up

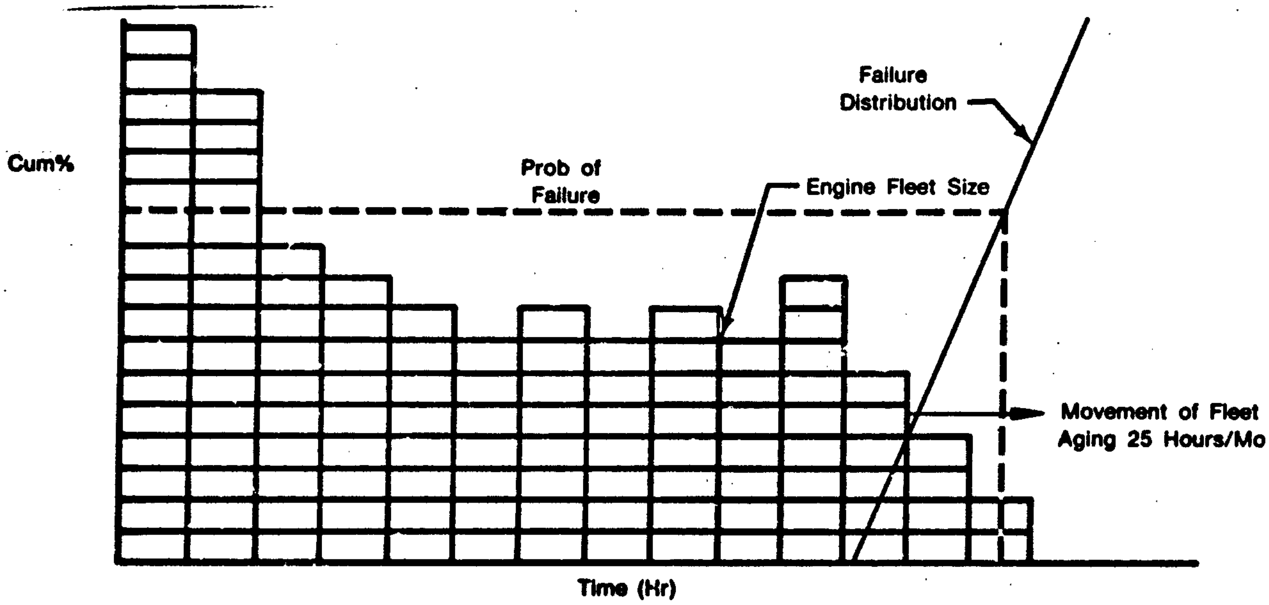


Figure 6.3. Risk Analysis

TABLE 6.1 PROJECTED PUMP FAILURES

Cumulative Failures	*Forecast Future Failures*
9.17	0.0 More Failures In 0 Months
12.26	3.12 More Failures In 1 Months
16.22	7.08 More Failures In 2 Months
21.14	11.98 More Failures In 3 Months
27.18	13.02 More Failures In 4 Months
34.50	25.33 More Failures In 5 Months
45.21	34.06 More Failures In 6 Months
53.44	44.27 More Failures In 7 Months
65.24	56.07 More Failures In 8 Months
78.65	69.48 More Failures In 9 Months
93.64	84.47 More Failures In 10 Months
110.1[illegible]	100.97 More Failures In 11 Months
128.01	118.85 More Failures In 12 Months
147.11	137.94 More Failures In 13 Months
167.21	158.05 More Failures In 14 Months
188.08	178.91 More Failures In 15 Months
206.40	200.24 More Failures In 16 Months
230.82	221.66 More Failures In 17 Months
251.89	242.73 More Failures In 18 Months
272.07	262.90 More Failures In 19 Months
290.75	281.58 More Failures In 20 Months
307.32	298.16 More Failures In 21 Months
321.27	312.11 More Failures In 22 Months
332.29	323.12 More Failures In 23 Months
340.34	331.18 More Failures In 24 Months

$\beta = 5.94$ $\eta = 462.2$ N = 353

Based on this analysis, it was recommended to Project Engineering that turbopumps No. 650 and up with more than 175 hours of time be replaced in the fleet. Fortunately, there were sufficient spare turbopumps to allow this to be accomplished without grounding aircraft. In addition, this would not have been possible without the knowledge of the relatively low risk between 0 time and 200 hours. This action was effective as there were no more field failures.

Laboratory analysis of the failed pumps indicated that the failure mode was caused by swelling of the plastic ball bearing cage to the extent that the balls would skid, causing the bearing to fail. Coordinating with the turbopump manufacturer, the bearing manufacturer, and the plastic manufacturer, a statistical factorial experiment was designed to determine the cause of the swelling of the plastic cages for corrective action.

6.6 FOUR MONTHS LATER — FINAL WEIBULL PLOT

Inspection of the turbopumps replaced in service (Number 650 and up with 175 hours or more) revealed 15 more bearings considered to be imminent failures. The addition of these failures to those originally seen in the field produced the final Weibull plot with 24 failures in a sample of 387 turbopumps (Figure 6.4). Note that the original three-failure curve is a good approximation of the final plot, the only difference being that the earlier curve had a steeper slope (10 rather than 4.6) as indicated on Figure 6.1. Although this slope difference sounds large, in fact, the inference from either curve would be substantially the same, that is, a rapid wearout problem. The second Weibull based on seven failures was also a good approximation of the final Weibull (Figure 6.4).

By this time the results of the statistically designed factorial experiment were available. It was found that a process change had been made in the manufacture of the plastic cage to reduce costs. The change resulted in cages of lower density. When these lower density cages were subjected to the combination of heat, fuel, and alcohol, the alcohol diffused through the plastic causing it to swell and crack. All such cages were removed from service. (Alcohol is a de-icing agent added to jet fuel.)

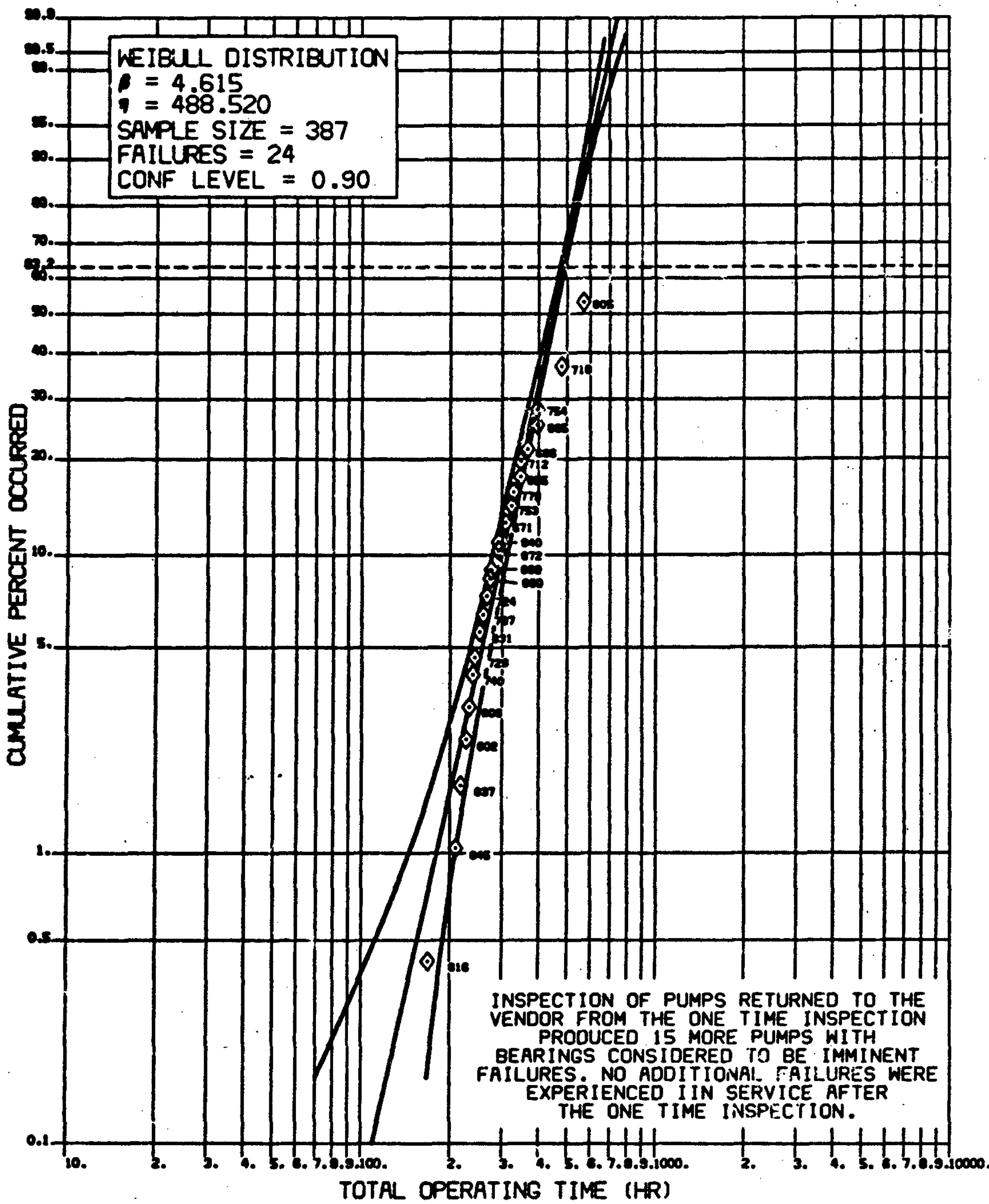
6.7 EXAMPLE 2: MAIN GEARBOX HOUSING CRACKS

The main gearbox housing on some engines developed cracks in the field. This type of crack would usually be discovered during an inspection for oil leaks. Cracked housings were being discovered at a rate of 1/20,000 hours of operating flight time. This was a ruggedly built gearbox housing, and it was questioned whether each crack was one of a kind or whether they were related events. Also, this identical gearbox was being introduced into a new aircraft, and it was questioned whether the same failure mode would appear in the new installation.

6.8 INFORMATION AVAILABLE FOR ANALYSIS

Once the field was alerted to cracked housings, a quick inspection revealed 27 cracked units. Of the 27, four housings were from the new aircraft.

At the outset, there was considerable discussion as to whether the data should be grouped together or a separate analysis should be completed for each aircraft type. Because of the different missions of the two aircraft, it was decided that separate analyses should be run. The Weibull analysis is presented in Figure 6.5 for both Aircraft 1 and Aircraft 2.



FD 271877

Figure 6.4 Weibull Plot for Augmentor Pump

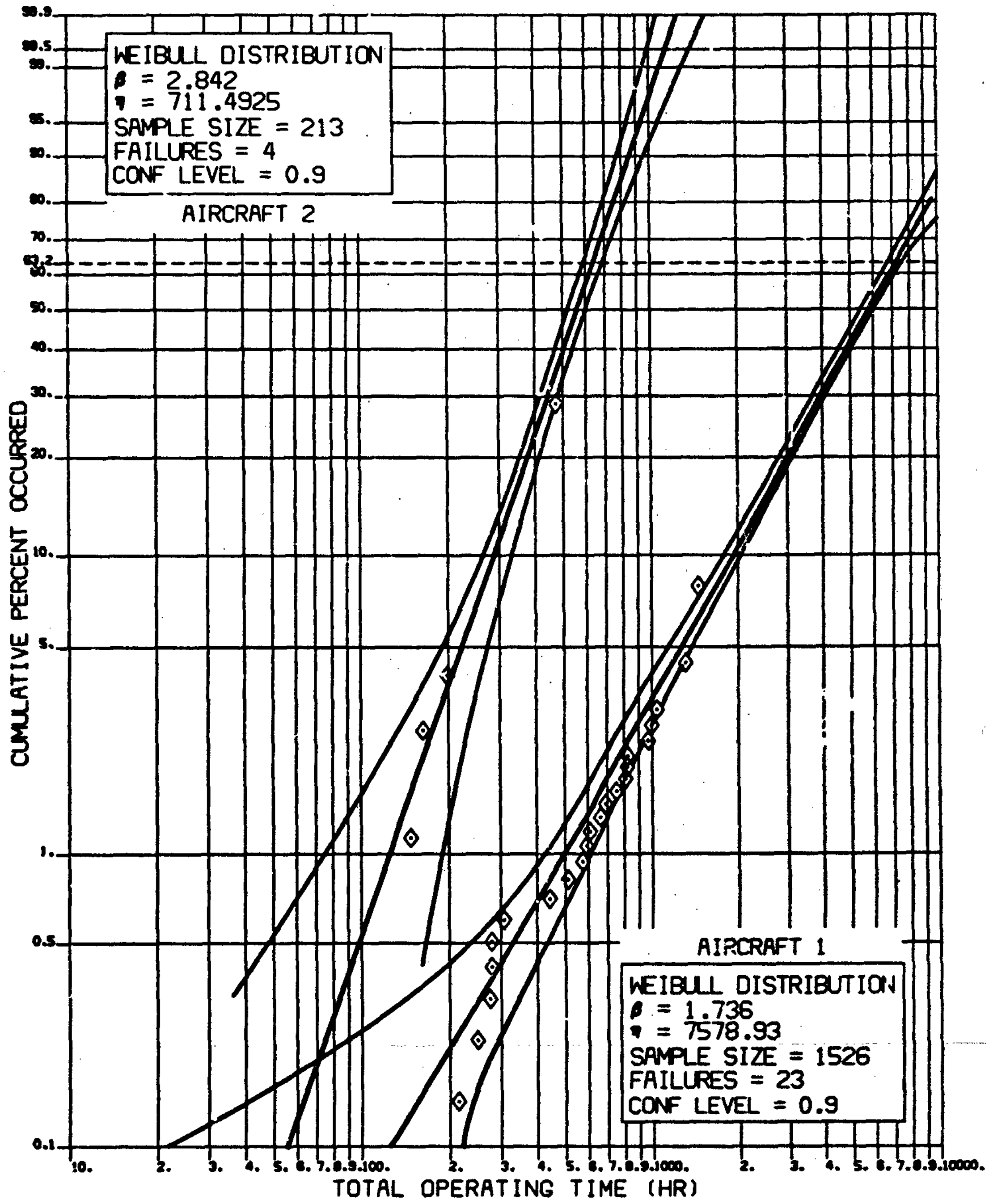


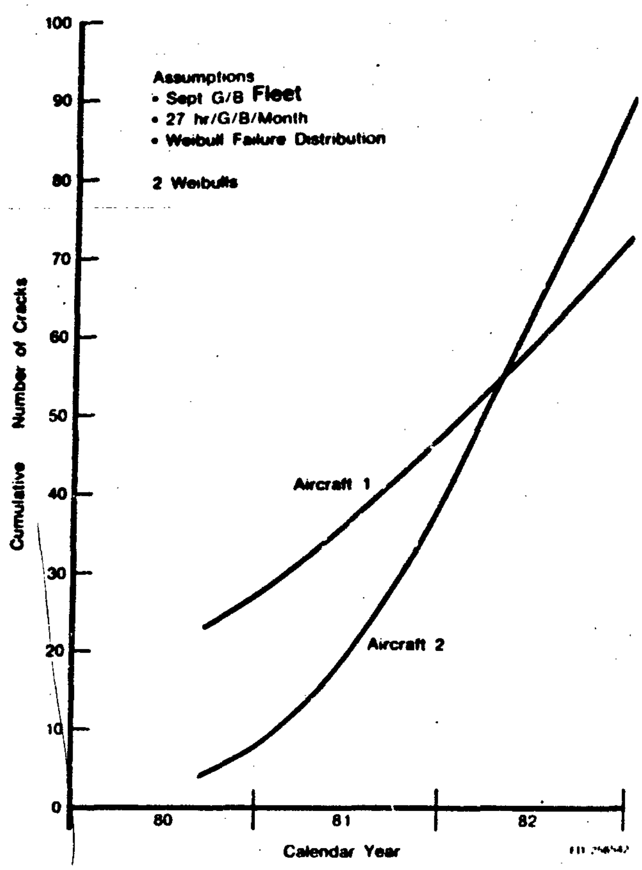
Figure 6.5. Main Gearbox Housing Cracks

At the time of the analysis, there were 23 cracked housings out of 1,526 in the Aircraft 1 fleet. From Aircraft 2, 4 out of 213 engines in the field were found with cracks in the housings. Both curves represented wearout modes, with Aircraft 2 having failures occurring earlier and at a faster rate (i.e., steeper slope).

One additional item which should be noted, especially in Aircraft 1, is that the data do not fall on a straight line. Ordinarily, data of this nature indicate that there may be more than one mode of failure. However, leaks and cracks of this type do not usually result in engine failure and are not discovered until an inspection is performed. The run time on the component at the time of the leak or crack is usually not well defined, and the Weibull is distorted by the clustering of events discovered at inspection. This would be especially true if the time between inspections is large but occurring at specific times. One way to correct for this type of analysis problem is to correct the data back to a common crack length. However, the correction factor often comes under question and the easiest way to avoid this argument is to present the data as they are obtained.

6.9 RISK ANALYSIS

A risk analysis for forecasting future failures was requested. This analysis used methods discussed in Chapter 3. The results of the analysis are presented in Figure 6.6 for both aircraft through 1982. It can be seen that Aircraft 2 has a lower characteristic life than Aircraft 1. This finding led to an investigation to determine if there were differences between the two aircraft which would account for the difference in characteristic life. Strain gages and vibration pickups were placed on gearboxes of each aircraft and data were obtained. It was concluded that Aircraft 2 was subject to more vibratory stresses which shortened the fatigue life of the gearbox. This would explain the steeper Weibull slope for Aircraft 2.



FD 256542

Figure 6.6. Cumulative Main Gearbox Housing Cracks

6.10 DETERMINING THE FIX

The fix for this problem was a fairly simple one. The cracking originated in the coverplate of the gearbox. The driving force was a coverplate resonance at certain engine speeds. The coverplate was redesigned not to resonate at these frequencies, thus eliminating the problem.

6.11 HOW GOOD WERE THE FORECASTS?

Because of the time lag from problem definition to the incorporation of a fix, additional failures may occur. This presents an opportunity to evaluate how well a forecast did and to monitor the effects of sample size on the Weibull parameters.

This problem was tracked for two years after the original analysis was complete. The findings of the initial analysis considered 23 cracks for Aircraft 1 and four cracks for Aircraft 2. The analysis was repeated several times for each aircraft as more information became available. Results of the follow-on analyses are presented in Table 6.2.

TABLE 6.2 FOLLOW-UP ANALYSIS RESULTS

Date	*No. of Engines*	*No. of Cracks*	β	η *(hours)*
Aircraft 1				
Original	1526	23	1.736	7578.3
12 mo. later	1609	41	1.782	6552.8
24 mo. later	1949	62	1.715	7038.0
Aircraft 2				
Original	213	4	2.842	711.5
12 mo. later	500	10	2.348	1533.7
24 mo. later	732	13	1.805	3604.8

With the large number of cracks associated with Aircraft 1, the Weibull is stable. However, Aircraft 2's Weibull has changed considerably. This is typical and is discussed extensively in Appendix F. The risk analysis reflects the type of conservatism that would be expected from the results of the initial analysis. The steepness in the slope would cause an overprediction of the expected number of cracks. From a risk viewpoint this could be considered as safety margin. However, before any action is taken to incorporate an engineering change to correct the problem, an analysis must also be performed to determine the cost effectiveness of the change.

6.12 EXAMPLE 3: OPPORTUNISTIC MAINTENANCE SCREENING INTERVALS

Often gas turbine engines are sent to the shop because of unexpected hardware failures or foreign object damage. Although the primary concern is the repair of the engine, the question also arises should the engine undergo its next scheduled maintenance while it is available in the shop. The answer is based on economic considerations and depends on how close the engine or its modules are to the next scheduled inspection.

For example, if an engine is in the shop for repair after 1340 cycles of operation and is due for a scheduled inspection at 1350 cycles (one cycle being equal to about 0.8 hour of engine flight time), there would be no question that it should be inspected before re-installation in the aircraft. If, however, the engine is in the shop at 1150 cycles, it is not so obvious that the 1350 cycle inspection should be performed. If the engine is in the shop at 500 cycles, it obviously should not be inspected (costly part replacements are involved). There is, therefore, a break-even point to be determined.

6.13 STRUCTURING THE PROBLEM

The trade to be made considers avoiding a scheduled engine removal versus the scrapping of parts whose life is not quite used up. The opportunistic use of an unscheduled engine removal to perform the scheduled inspection and replacement of life-limited parts not only allows saving the labor involved in engine removal and replacement but also allows the purchase of fewer "pipeline" spare engines and modules. These savings are weighed against the added costs incurred by replacing parts early.

6.14 FINDING THE OPTIMUM INTERVAL

Monte Carlo simulation is the method preferred for evaluating the range of opportunistic maintenance intervals. The U.S. Air Force has such a simulator which (with some modification) could be used for determining the optimum opportunistic maintenance interval. The simulator is structured to perform scheduled maintenance whenever the Monte Carlo process selects an unscheduled engine removal which falls within a predefined screening interval. The process is repeated for various screening intervals, and the resultant total support cost is plotted against the selected screening interval to determine the optimum. Figure 6.7 is an illustration of this procedure.

The Weibulls are used to describe each of the engine modules' major failure modes (reason for unscheduled removal). Where improvements have been incorporated, the Weibulls are adjusted to reflect the improvements. Only with a valid representation of the way in which each removal cause varies with time could a realistic assessment be made.

The simulation analysis was performed and an opportunistic maintenance interval of 300 cycles was determined. This provided the Air Force with an economic decision criterion for performing scheduled maintenance.

6.15 EXAMPLE 4: SUPPORT COST MODEL

The support cost model uses a Monte Carlo approach to simulate the interaction of scheduled and unscheduled maintenance events. The unscheduled events are entered in the form of Weibull curves relating event, probability, and time. Scheduled events are entered at specific times. A screening interval is input to define a time period during which scheduled events can be precipitated by unscheduled opportunities. (See Figure 6.8.) Labor and material costs are input for each event. The model selects corresponding labor and material costs for each event and compiles totals for the number of events and for labor and material costs by report period (year). Totals are divided by the number of flight hours for the report period to derive rates per flight hour.

The model makes a predesignated number of passes through the life cycle (20 years) and reports the average of the passes by report period. The number of events per year can therefore appear as a non-integer.

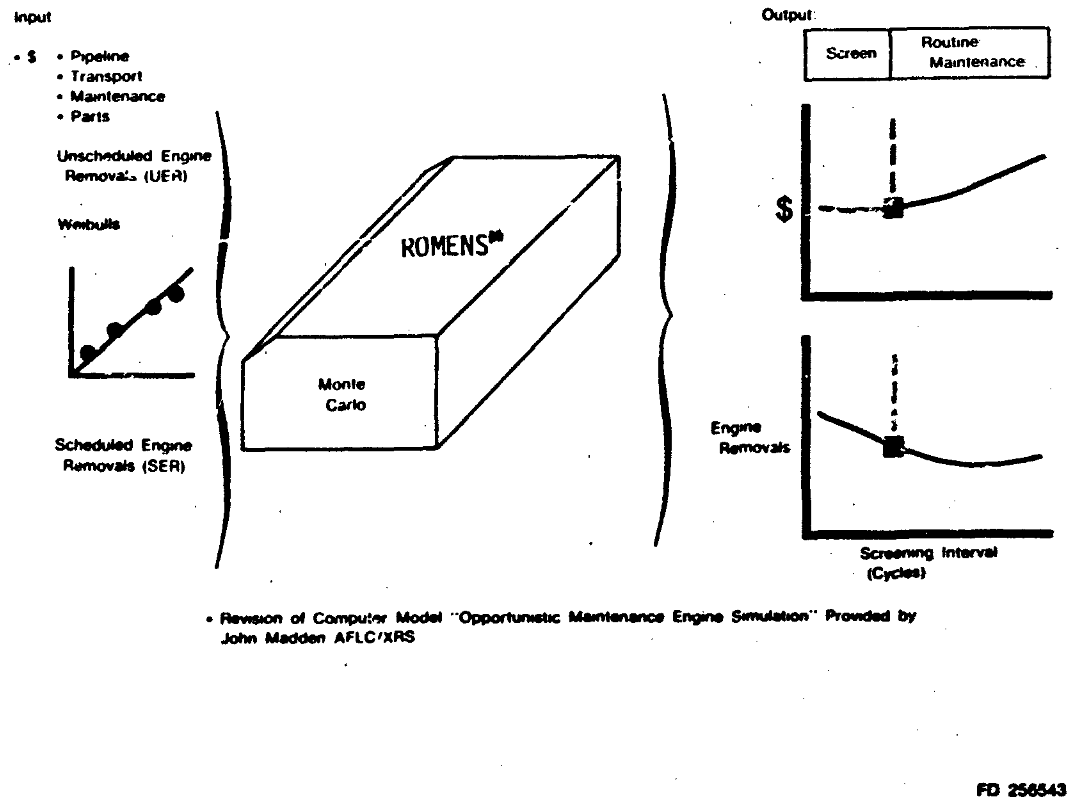


Figure 6.7. Approach To Optimizing Scheduled Maintenance

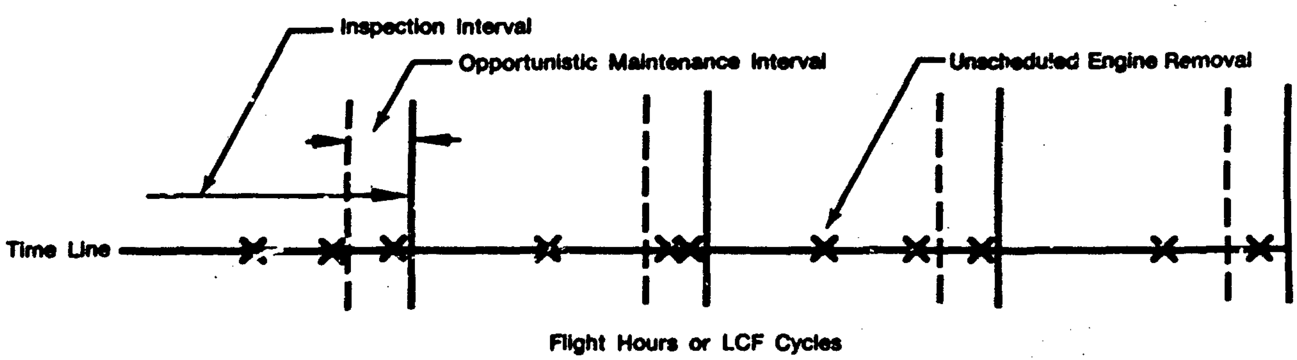


Figure 6.8. Scheduled and Unscheduled Maintenance Interaction

Unscheduled maintenance events, input as Weibulls, are of four basic types: 1) unscheduled engine removals (UER's), 2) unscheduled module removals which are coincidental with an engine removal (coincidentals), 3) unscheduled part removals which are coincidental with a module removal (part coincidentals), and 4) installed maintenance events. As stated above, each of these event inputs is accompanied by a corresponding labor and material cost. It is also accompanied by factors which designate those percentages of events which precipitate engine or module depot visits and demands for spare modules.

Scheduled event input is also accompanied by labor and material costs per event. Material is input as a total cost of parts involved in the inspection along with a percentage to be scrapped at each event. This scrap rate can vary among the events of a particular sequence.

The input is then a combination of Weibulls, scheduled intervals, material and labor costs, depot visit factors, and supply system demand factors. Output is reported at the module (failure mode) level, by report period (year), in terms of total quantities and rates per flight hour. Parameters reported include engine removal and depot visits, module removals and depot visits, module demands, labor and material costs broken down by depot and base, and scheduled vs unscheduled maintenance for each report period and for the total life cycle.

6.16 ROLE OF THE WEIBULL

Unscheduled engine maintenance, as indicated above, is driven by both scheduled and unscheduled events. The unscheduled events are caused by some failure modes that occur randomly and others that exhibit wearout characteristics, i.e., an increasing failure rate. The Weibull is the most convenient method of introducing these increasing rates into the model.

The Weibull is described by only two parameters, the characteristic life, η, and the slope, β. Figure 6.9 illustrates the use of Weibulls with $\beta > 1$ for life limited parts and $\beta = 1$ for randomly distributed failure modes. Infant mortality, although seldom encountered in an operational engine, can also be simulated by $\beta < 1$.

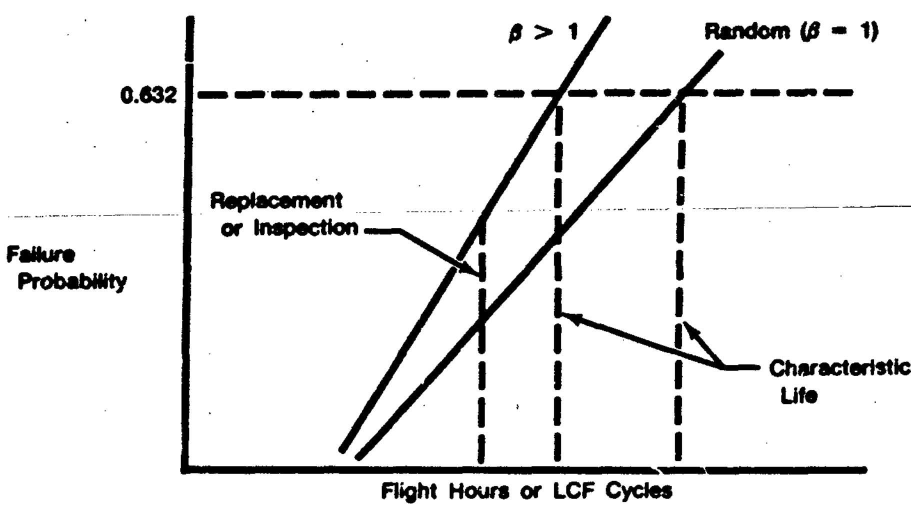


Figure 6.9. Unscheduled Maintenance Input via Weibulls

6.17 EXAMPLE 5: VANE AND CASE FIELD CRACKS

Cracks found in the 12th stage vane and case of a high pressure compressor precipitated this study. There was concern of case rupture if the cracks grew large enough to weaken the structure. The major questions were:

(1) How will the problem affect engines?
(2) What can be done to fix the problem?
(3) Can the problem be detected through inspection?
(4) What recommendations should be made to the Air Force?

6.18 RESOLVING THE QUESTIONS

Seven cracked cases were identified. The cracks were all of different lengths but shorter than considered critical. One question which often arises from this type of analysis is whether to normalize the times to a constant crack length. It was decided to proceed without the corrections on the times and construct a Weibull from the available information. Figure 6.10 is the Weibull based on seven cracked cases.

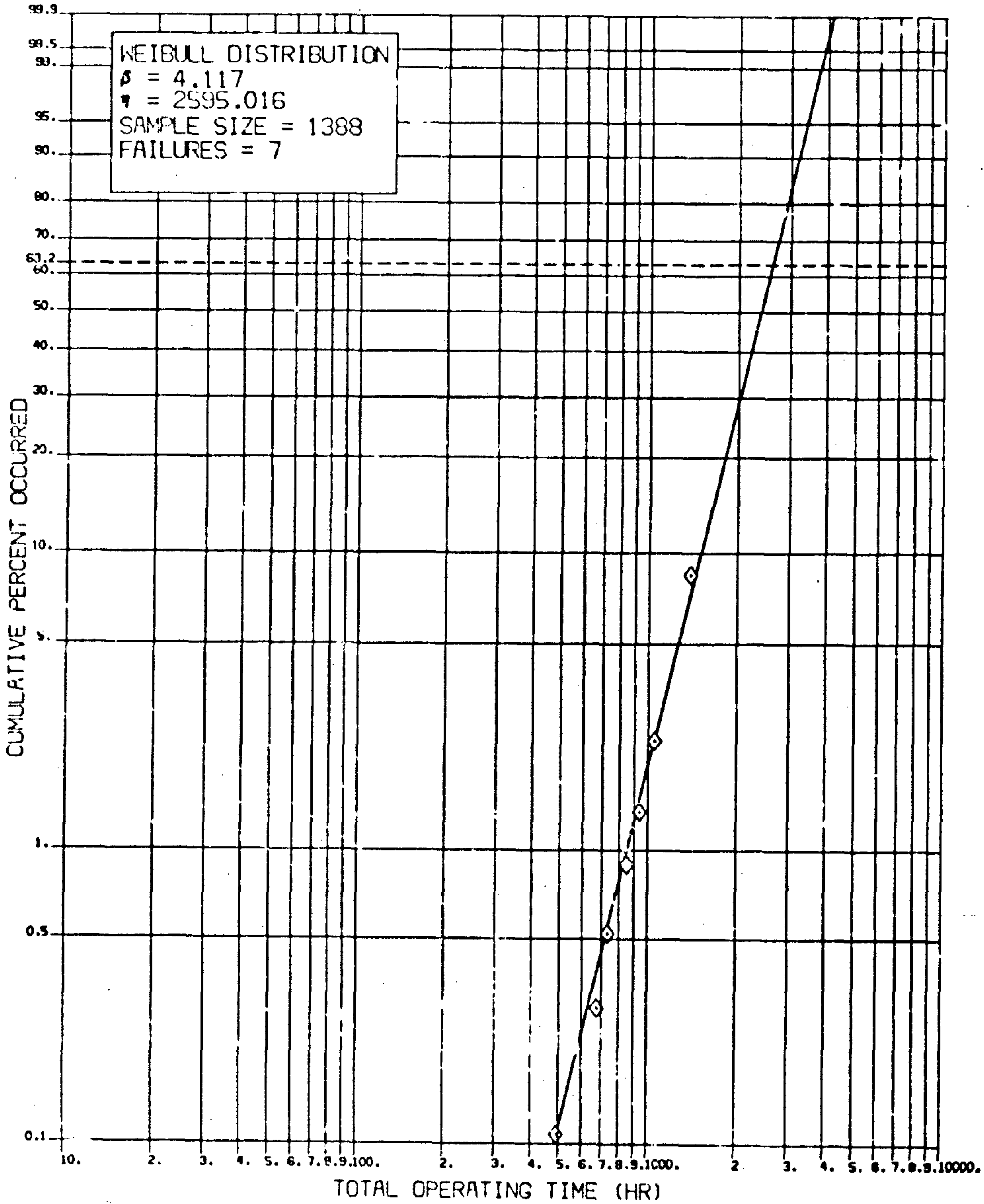
The rate at which the fleet would run into this problem was then examined. It was assumed that all engines were susceptible and that the crack could be detected upon inspection. These cases would be repaired by welding, and the units would be placed back into operation. This inspection and repair could be continued until a fix was in place. The more permanent fix was to hardcoat the area of the cracking with a plasma spray. It was also assumed that the fleet would accumulate an average of 27 hours per month on each engine.

The engine would normally undergo inspection at 1350 cycles. This is equivalent to about 1080 hours of engine operation.

With these assumptions and assuming that the hardcoat fix would be applied to all new engines, the number of unscheduled engine removals due to this problem was projected using methods described in Chapter 3. The results are illustrated in Figure 6.11. The forecast of 10 or more engines developing cracks by the end of the first year and the total reaching about 40 by the end of the following year resulted in implementation of the hardcoat fix.

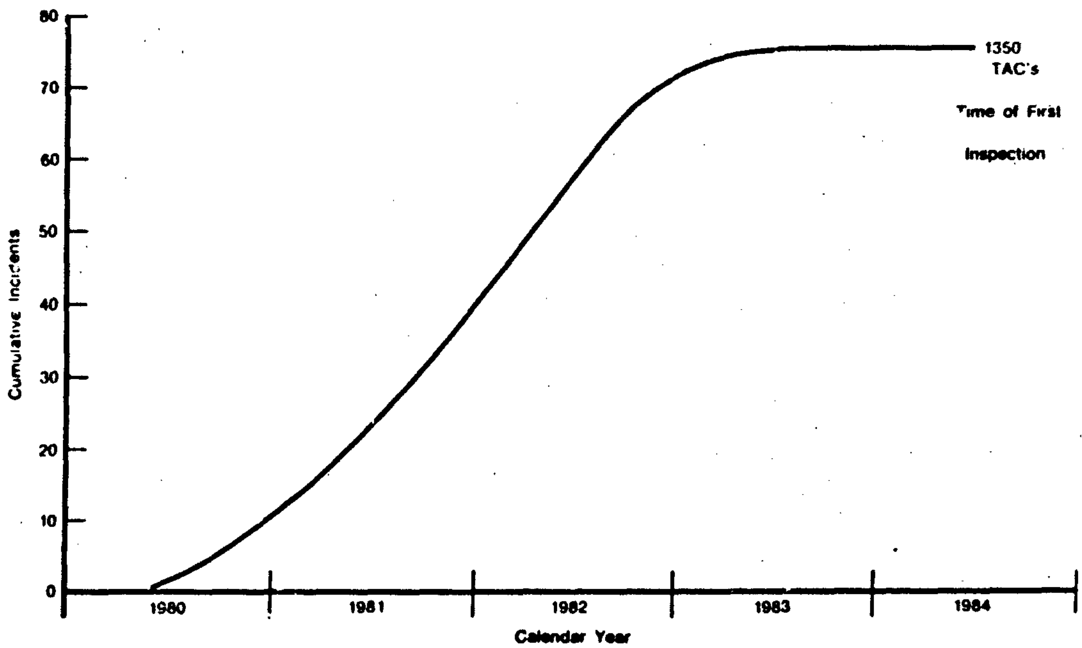
6.19 CONCLUDING REMARKS

The plasma spray hardcoat has been incorporated into production units. In addition, as old units are received for their normal overhaul, hardcoating is applied to these units as well. At this writing, a total of only 15 engines have been identified with cracks over the critical limit where it could be said from the forecast that 40 additional engines would have been expected without the fix. The quick action by the Air Force to implement the fix resulted in correcting the condition in the field.



FD 271879

Figure 6.10. 12th Vane and Case Cracking



FD 256547

Figure 6.11. Expected UER's Due to 12th Vane and Case Cracking

CHAPTER 7

CONFIDENCE LIMITS AND OTHER ASPECTS OF THE WEIBULL

7.1 FOREWORD

Now that some familiarity has been developed with the Weibull distribution and its application in risk analysis and life testing, further applications will be discussed. First among these will be confidence intervals about the Weibull parameters β and η and about the Weibull line. Secondly, special applications in risk analysis will be discussed, namely Weibull "Thorndike" charts. The next topic to be discussed will be shifting Weibulls in the case of insufficient information about the underlying population. Lastly, other options available when the Weibull distribution may not fit the failure data will be discussed.

7.2 CONFIDENCE INTERVALS

Confidence intervals are measurements of precision in estimating a parameter. A confidence interval around an unknown parameter is an interval of numbers derived from sample data that almost surely contains the parameter. The confidence level, usually 90% or higher, is the frequency with which the interval calculation method could be expected to contain the parameter if there were repeated applications of the method.

7.3 CONFIDENCE INTERVALS FOR β AND η

Often it is of interest to determine how far from the "true" value an estimate of β or η might deviate. For example, if the times to failure of every bearing ever made and every bearing to be made in the future were known, it would be possible to calculate β and η exactly. But, of course, this is never the case; only a sample of bearings is available. The question is: how much variation can be expected in the estimates of β and η ($\hat{\beta}$ and $\hat{\eta}$) from one sample size to the next? If this variation is small, then the particular sample will yield estimates close to the true values.

The problem involving censoring with very few failures is not dealt with here. Reference [1] is recommended for this situation. However, for large, complete (no suspensions) samples of size n, the confidence intervals for β and η can be approximated by equations (7.1) and (7.2), respectively.

$$\hat{\beta}\exp\left(\frac{-0.78Z_{\alpha/2}}{\sqrt{n}}\right) \leq \beta \leq \hat{\beta}\exp\left(\frac{0.78Z_{\alpha/2}}{\sqrt{n}}\right) \tag{7.1}$$

$$\hat{\eta}\exp\left(\frac{-1.05Z_{\alpha/2}}{\hat{\beta}\sqrt{n}}\right) \leq \eta \leq \hat{\eta}\exp\left(\frac{1.05Z_{\alpha/2}}{\hat{\beta}\sqrt{n}}\right), \tag{7.2}$$

where $Z_{\alpha/2}$, the upper $\alpha/2$ point of the standard normal distribution, depends on what confidence level is chosen. Table 7.1 gives $Z_{\alpha/2}$ for various (usual) confidence levels.

TABLE 7.1. CONFIDENCE LEVELS

Confidence Level	$Z_{\alpha/2}$
99%	2.576
95%	1.960
90%	1.645

[1] *Applied Life Data Analysis*, Nelson, 1982.

These confidence intervals are only approximate since: (1) the estimates of β and η used are linear regression estimates (from a theoretical standpoint maximum likelihood estimates would be required, see Appendix D), (2) these estimates are only approximately normally distributed.

Example 7.1

Figure 7.1 shows a fitted Weibull distribution with 45 failures and no suspensions. A 90% confidence interval for β is desired. The relevant information is as follows:

$n = 45$
$\hat{\beta} = 1.84$
Confidence level = 90%
$Z_{\alpha/2} = 1.645$ (from Table 7.1)

Substituting into equation 7.1,

$$1.84 \exp\left(\frac{-0.78(1.645)}{\sqrt{45}}\right) \leq \beta \leq 1.84 \exp\left(\frac{0.78(1.645)}{\sqrt{45}}\right)$$

which reduces to $1.52 \leq \beta \leq 2.23$

Example 7.2

Using the Weibull from Figure 7.1, what is a 90% confidence interval for η? The relevant information is:

$n = 45$
$\hat{\eta} = 958.88$, $\hat{\beta} = 1.84$
Confidence level = 90%
$Z_{\alpha/2} = 1.645$ (from Table 7.1)

Substituting into equation 7.2,

$$958.88 \exp\left(\frac{-1.05(1.645)}{1.84\sqrt{45}}\right) \leq \eta \leq 958.88 \exp\left(\frac{1.05(1.645)}{1.84\sqrt{45}}\right)$$

or, $833.7 \leq \eta \leq 1102.9$

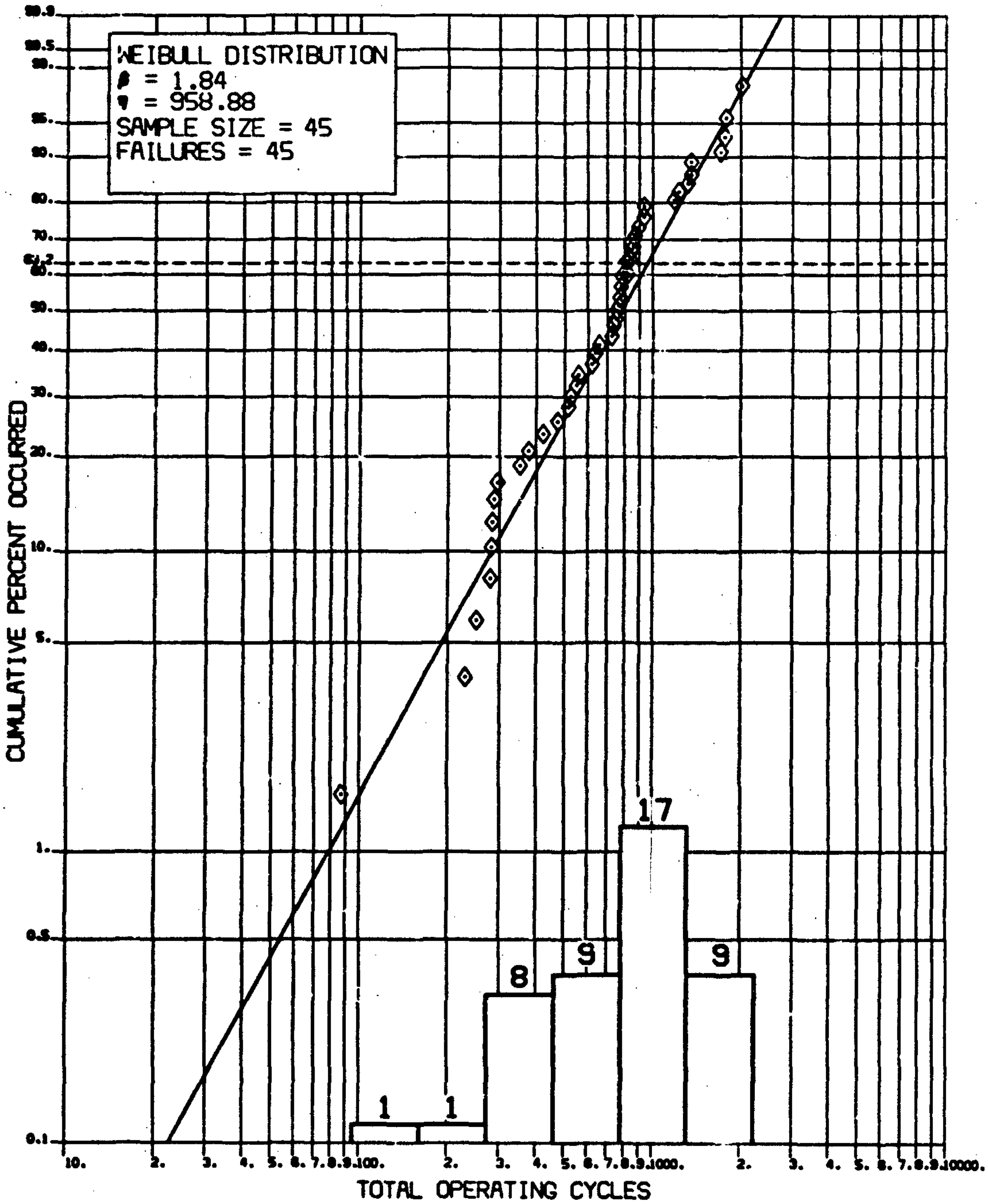


Figure 7.1. Weibull Test Case

7.4 CONFIDENCE INTERVALS FOR RELIABILITY

Another problem that appears in Weibull analyses is that of obtaining confidence intervals for the reliability at a given point in time. The reliability at the point t' is the probability of a life of at least t' units, and will be denoted by R(t'). Again assume a large sample with no censored observations.

The procedure is as follows:

1. Compute $\hat{u} = (\ell n(t) - \ell n(\hat{\eta}))\hat{\beta}$ (7.3a)

2. Compute $\mathrm{Var}(\hat{u}) = [\,1.168 + 1.10\,(\hat{u}^2) - 0.1913\,\hat{u}\,] \cdot \frac{1}{n}$ (7.3b)

3. Compute $u_1 = \hat{u} - Z_{\alpha/2}\,|\mathrm{Var}(\hat{u})|^{1/2}$ (7.3c)

   $u_2 = \hat{u} + Z_{\alpha/2}\,|\mathrm{Var}(\hat{u})|^{1/2}$ ($Z_{\alpha/2}$ from Table 7.1)

4. Then the confidence interval is: $\exp(-\exp(u_2)) \leq R(t') \leq \exp(-\exp(u_1))$ (7.3d)

Example 7.3

Again using Figure 7.1, a 90% confidence interval for the reliability at 700 cycles is desired. The step-by-step procedure follows:

1. $\hat{u} = (\ell n(700) - \ell n(958.88))\,1.84 = -0.579$

2. $\mathrm{Var}(\hat{u}) = [\,1.168 + (-0.579)^2(1.10) - 0.1913\,(-0.579)]\,\frac{1}{45} = 0.036$

3. $u_1 = -0.579 - (1.645)\,\sqrt{0.036} = -0.890$

   $u_2 = -0.579 + (1.645)\,\sqrt{0.036} = -0.268$

4. $\exp(-\exp(-0.268)) \leq R(700) \leq \exp(-\exp(-0.890))$

   Therefore, the confidence interval is $0.465 \leq R(700) \leq 0.663$

7.5 CONFIDENCE INTERVALS ABOUT A FAILURE TIME

Engineers are often interested in a confidence interval for the time associated with a given failure. This confidence interval can be approximated by equation (7.4). Ninety percent confidence intervals will be assumed for all confidence intervals about the Weibull line in this section.

$$t_i,\,0.05 = \eta\left[\ell n\frac{1}{1 - F_i(0.05)}\right]^{1/\beta},\quad t_i,\,0.95 = \eta\left[\ell n\frac{1}{1 - F_i(0.95)}\right]^{1/\beta} \qquad (7.4)$$

where t_i, 0.05 and t_i, 0.95 are the failure times associated with the i[th] failure and F_i (0.05) is the 5% rank associated with the i[th] failure, while F_i (0.95) is the 95% rank associated with the i[th] failure. Tables for F_i (0.05) and F_i (0.95) are in Appendix B.

Example 7.4

Suppose we are given the Weibull in Figure 7.2, $\beta = 2.0$, $\eta = 100$ hours, produced from 10 failures. The calculation procedure for the confidence interval about the first failure (26.1 hours) follows:

$F_1(0.05) = 0.005$ (from Appendix B)

$$t_1, 0.05 = 100\left[\ln\frac{1}{1-0.005}\right]^{1/2} = 100\,|0.00501|^{1/2}$$

$$= 7.08 \text{ hours}$$

$F_1(0.95) = 0.258$ (from Appendix B)

$$t_1, 0.95 = 100\left[\ln\frac{1}{1-0.258}\right]^{1/2} = 100\,|0.2984|^{1/2}$$

$$= 54.63 \text{ hours}$$

7.6 CONFIDENCE BANDS ON THE WEIBULL LINE

In Section 7.4 the confidence bands about a single reliability were calculated. Simultaneous confidence bands can also be placed on the Weibull distribution for complete samples. Reference[2] contains the basic information for their construction. The results in Reference[2] have been extended to the Weibull Distribution. Equation (7.5) together with Table 7.2 can be used to calculate 90% confidence bands about the Weibull Distribution.

$$(F(x) - K(n), F(x) + K(n)),$$

where

$$F(x) = 1 - e^{-(x/\eta)^{\beta}}, \tag{7.5}$$

and F(x) is the estimate obtained by substituting maximum likelihood estimates for the parameters.

[2] "An Approach to the Construction of Parametric Confidence Bands on Cumulative Distribution Functions," Srinivasan and Kanofsky, *Biometrika*, Vol. 59, 3, 1972.

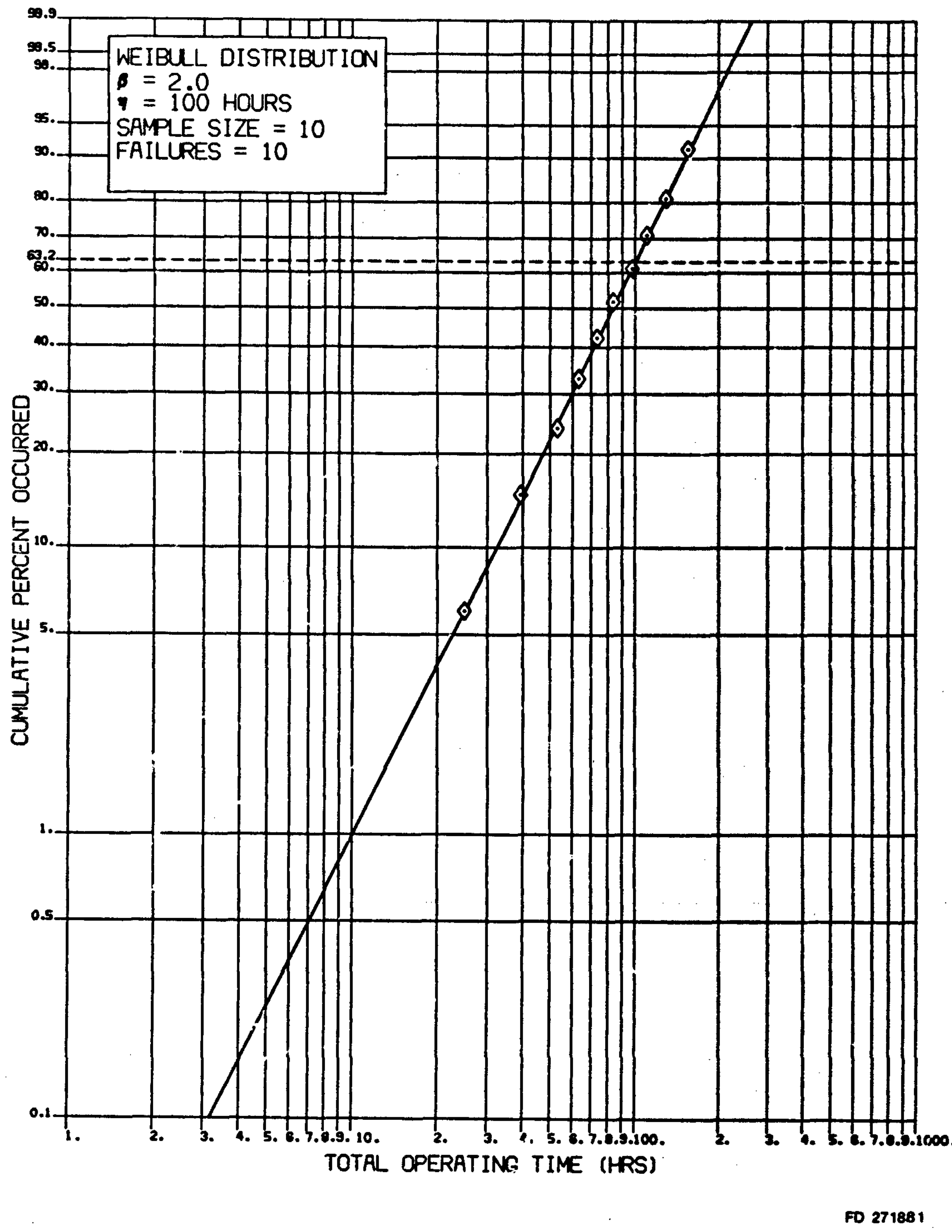


Figure 7.2. Weibill Plot Where β = 2.0 and η = 100 for 10 Failures

TABLE 7.2. CONFIDENCE BOUNDS ON THE WEIBULL LINE

Sample Size (n)	*K(n)*
3	0.540
4	0.420
5	0.380
6	0.338
7	0.307
8	0.284
9	0.269
10	0.246
11	0.237
12	0.222
13	0.213
14	0.204
15	0.197
20	0.169
25	0.152
30	0.141
35	0.125
40	0.119
45	0.117
50	0.106
75	0.086
100	0.074

Example 7.5

Consider the Weibull in Figure 7.3, $\beta = 2.0$ and $\eta = 2000$ hours, with a sample size of 7. From Table 7.2 the critical value of $K(7) = 0.307$. Therefore,

$$1 - e^{-(X/2000)^{2.0}} - 0.307 \leq F(x) \leq 1 - e^{-(X/2000)^{2.0}} + 0.307$$

for all x, $0 \leq x < \infty$, with 90% confidence.

These bands are illustrated in Figure 7.3.

7.7 WEIBULL "THORNDIKE" CHARTS

A graphical method often used to determine the cumulative probabilities of the Poisson distribution was named for F. Thorndike and is illustrated in Figure 7.4.

A random variable x has a Poisson distribution with a parameter μ if $P(X = x) = \exp(-\mu)\, \mu^x/x!$, $x = 0, 1, 2, 3, \ldots$). (The Poisson distribution also arises as the limiting form of the binomial when the sample size becomes large.) As an illustration, suppose it is necessary to make the statement: the expected number of occurrences is 3.0, and the actual number of occurrences will be between x and y with 0.90 probability. To find x and y, use the Thorndike chart in Figure 7.4 enter the x — axis at 3.0 and read up to the point where 3.0 intersects the 0.05 and 0.95 lines extending from the y - axis. The values for "C" are found to be about 0 and 6.0 respectively. Therefore, with probability 0.9, if the expected number from a Poisson distribution is 3.0, less than 6.0 will occur.

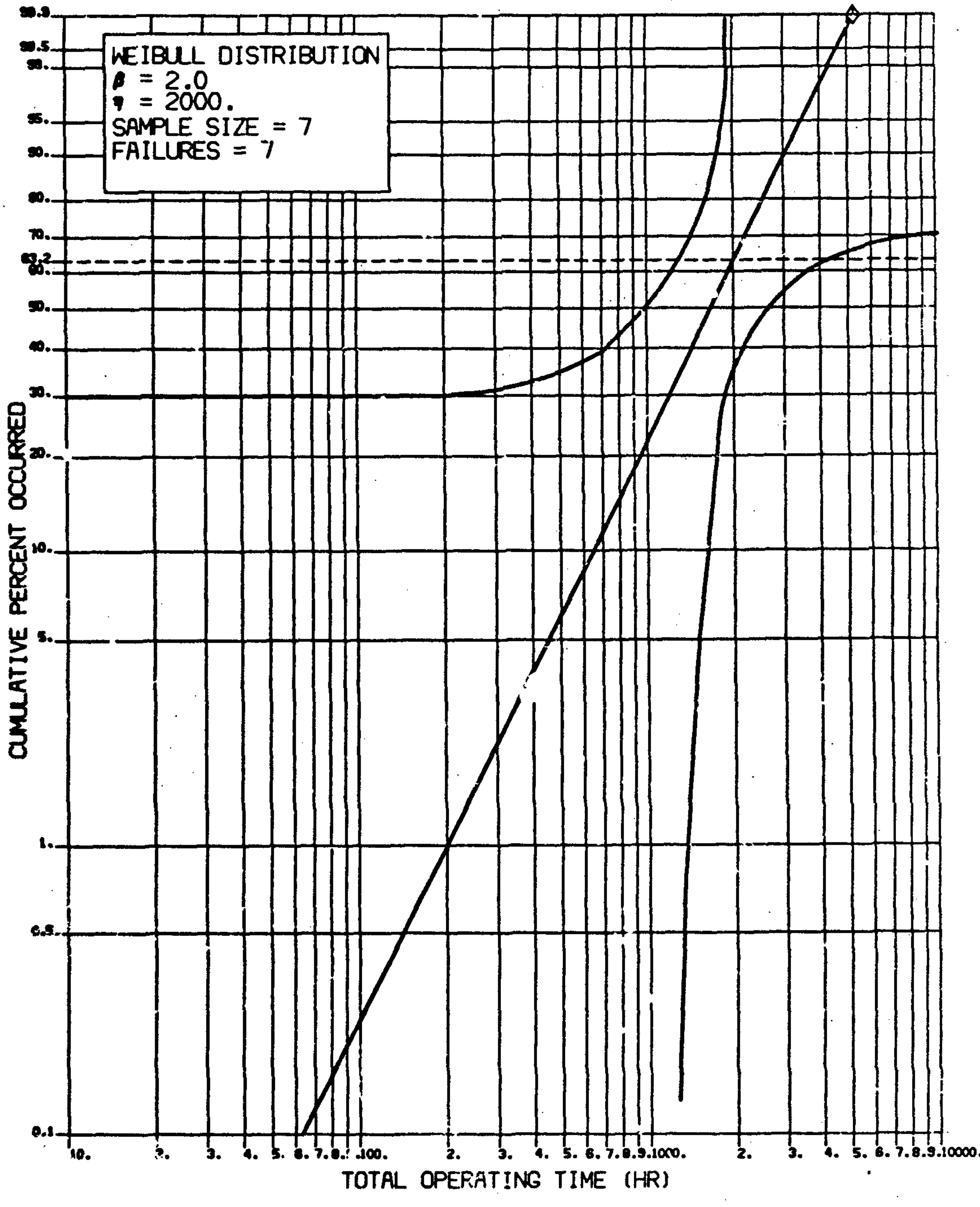


Figure 7.3. Example of Confidence Bands on a Weibull Line

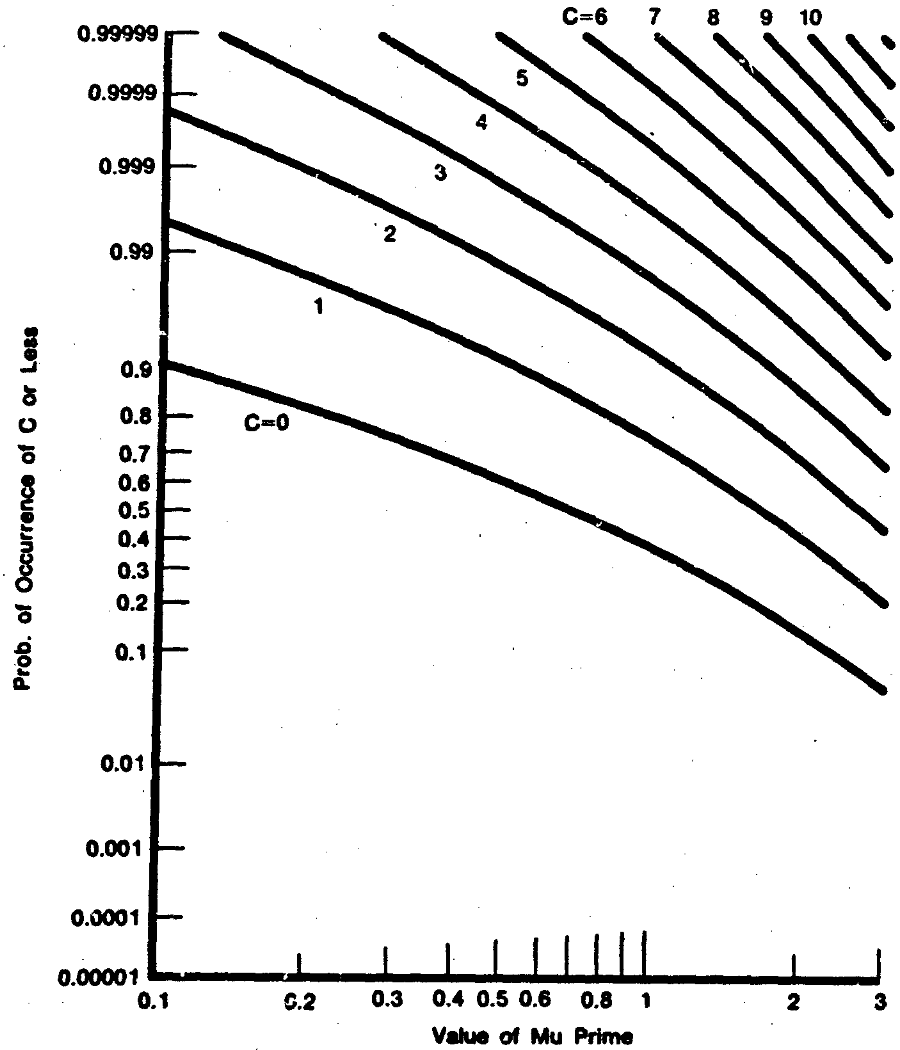


Figure 7.4. Cumulative Sums of Poisson (Thorndike Chart)

In this same way a graphical technique similar to the Thorndike chart has been developed for the Weibull distribution. These charts give the probability of having "C" or fewer failures by any given time. They can also be used to place bounds about the number of failures occurring by a given time.

Figures 7.5 through 7.12 are Weibull Thorndike charts for β's of 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, and 5.0. To use these charts, determine the time (t) of interest (possibly the inspection time), calculate t/η (η is the characteristic life from the Weibull), enter the x - axis of the chart with the closest β to the one of interest, and then read the probability of having "C" or fewer failures. Several examples of this technique and other uses follow.

The usefulness of this information can arise, for example, when the inspection interval is two or more times the characteristic life of the Weibull failure mode of a part. When this happens, the part fails, is replaced (made "good as new"), fails again, is replaced again, etc. How often can this process continue? The Weibull Thorndike charts answer this question.

Example 7.6

Given Weibull parameters β = 1.5 and η = 3000 hours, the probability of having three or fewer failures per unit by 6000 hours is to be calculated.

In this case, t/η = 6000/3000 = 2.0, and β = 1.5, so using Figure 7.7, enter the x - axis at t/η = 2.0 and proceed up to the point where the line "C = 3" is intersected. Then the probability of observing 3 or fewer failures can be read from the y - axis as 0.93.

Example 7.7

Suppose β = 1.5 and now t/η = 3.0. A 0.90 probability band can be placed about the number of failures occurring by t/η = 3.0.

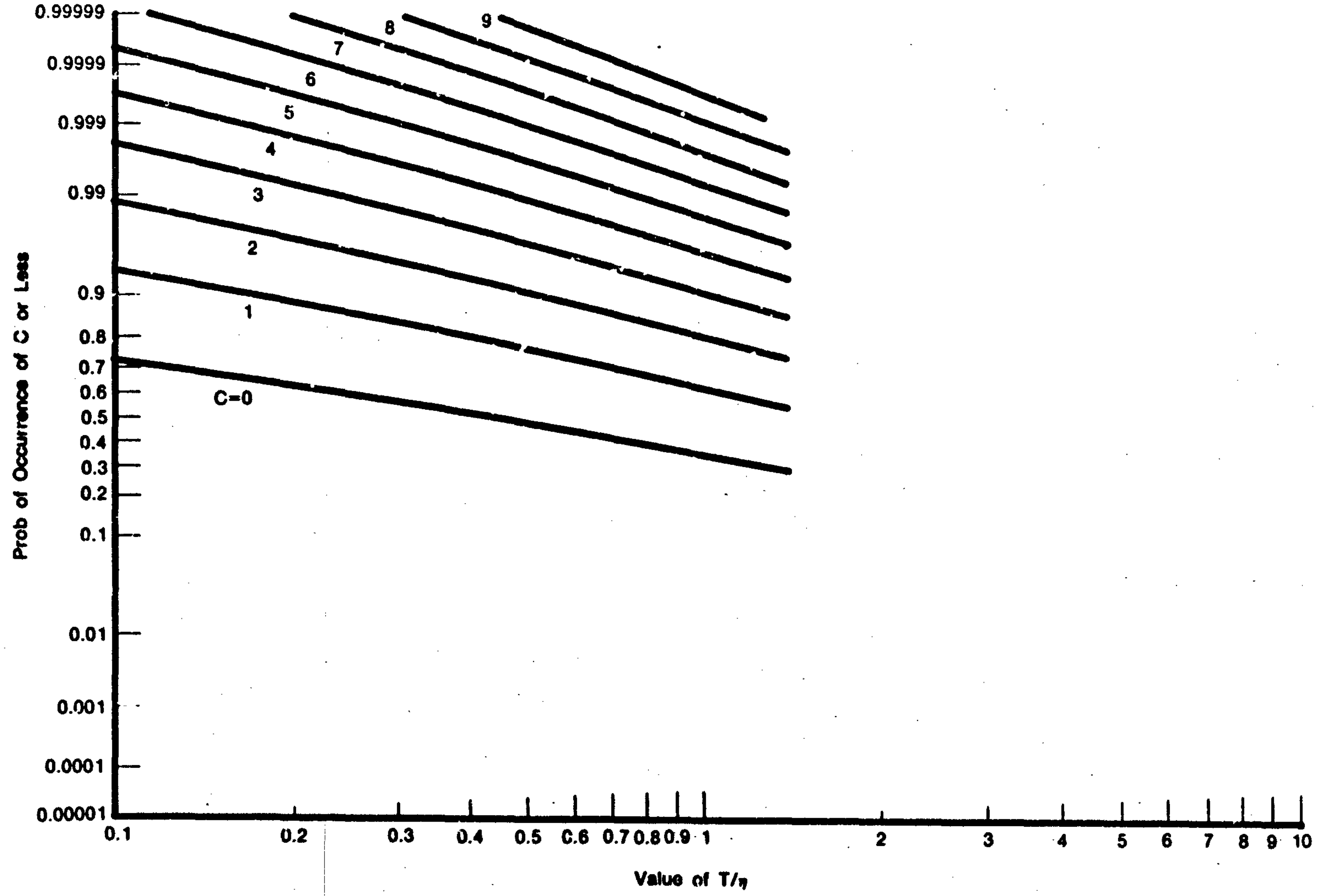
Again using Figure 7.7, enter the x - axis at 3.0 and proceed to find the "C" values where 0.05 and 0.95 probabilities intersect. This yields 1 and 5, respectively.

Example 7.8

In spare parts provisioning, suppose the number of spare parts to be provided are required for a part having a β = 3.0 and an inspection time/characteristic life ratio (t/η) = 2.0. The manager wishes to be 90% confident that he will not run out of parts. Using Figure 7.10, entering the x - axis at 2.0 and proceeding to the point where 0.9 on the y - axis intersects the "C" lines, no more than two spare parts are needed per delivered part.

Example 7.9

A new design rotor bearing has been tested for 22,000 hours. The current rotor bearing has a limiting failure mode whose β = 1.7 and η = 4,937 hours. Six failures have been observed in the test of the new design due to this mode. Is this unusual? With a t/η ratio = 22,000/4,937 = 4.45, entering the Weibull Thorndike chart for β = 1.5 (Figure 7.7), the probability of having six or more failures is approximately 1.0 - 0.90 = 0.1. Entering the Weibull Thorndike chart for β = 2.0 (Figure 7.8) the probability of having six or more failures is approximately 1.0 - 0.92 = 0.08. Therefore, it can be stated that the probability of observing six failures by this time in the redesigned rotor bearing is from 0.08 to 0.10. Hence the redesigned rotor bearing is not as good as the current bearing.



FD 258456

Figure 7.5. Weibull Thorndike Chart for β = 0.50

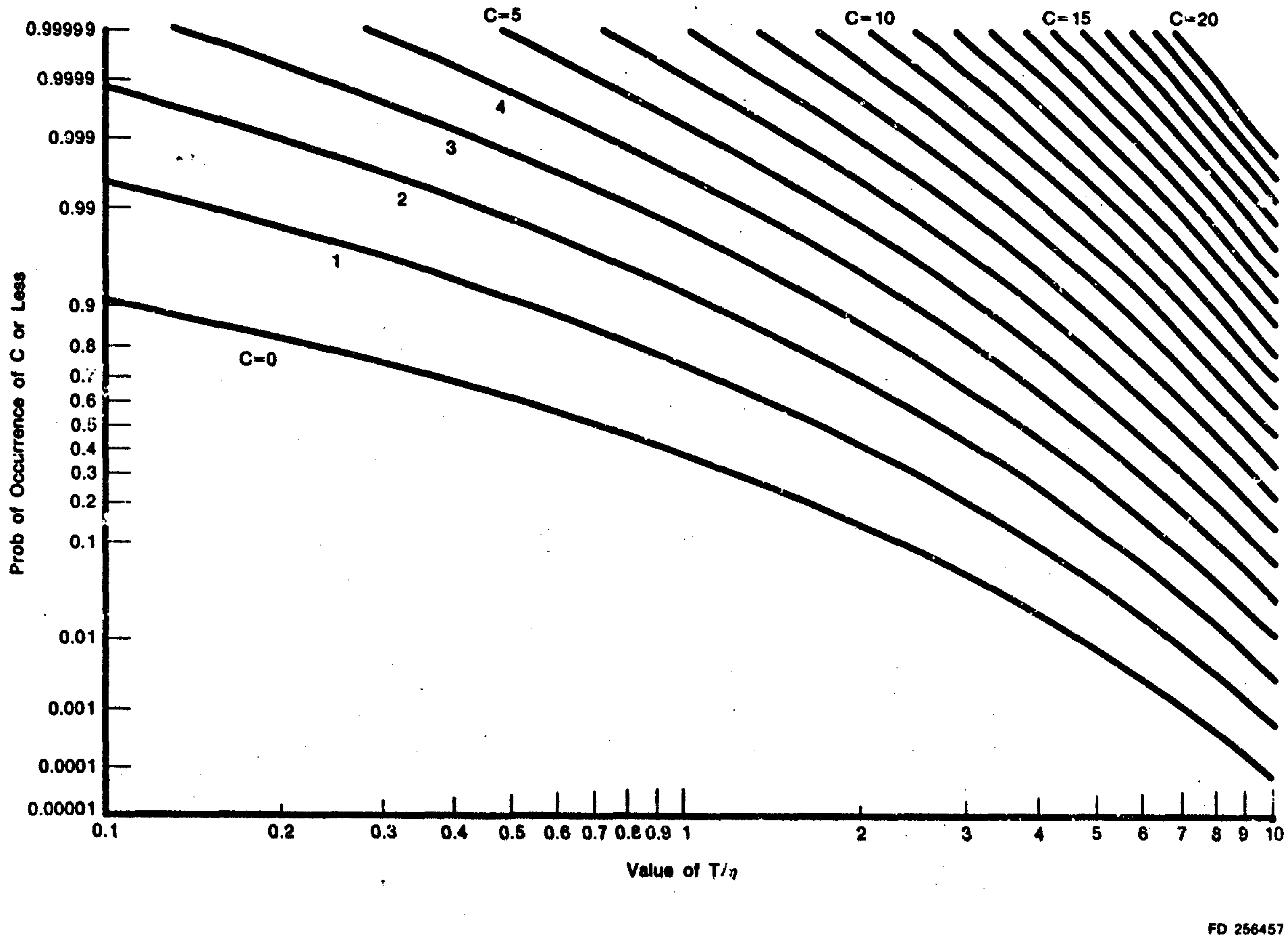
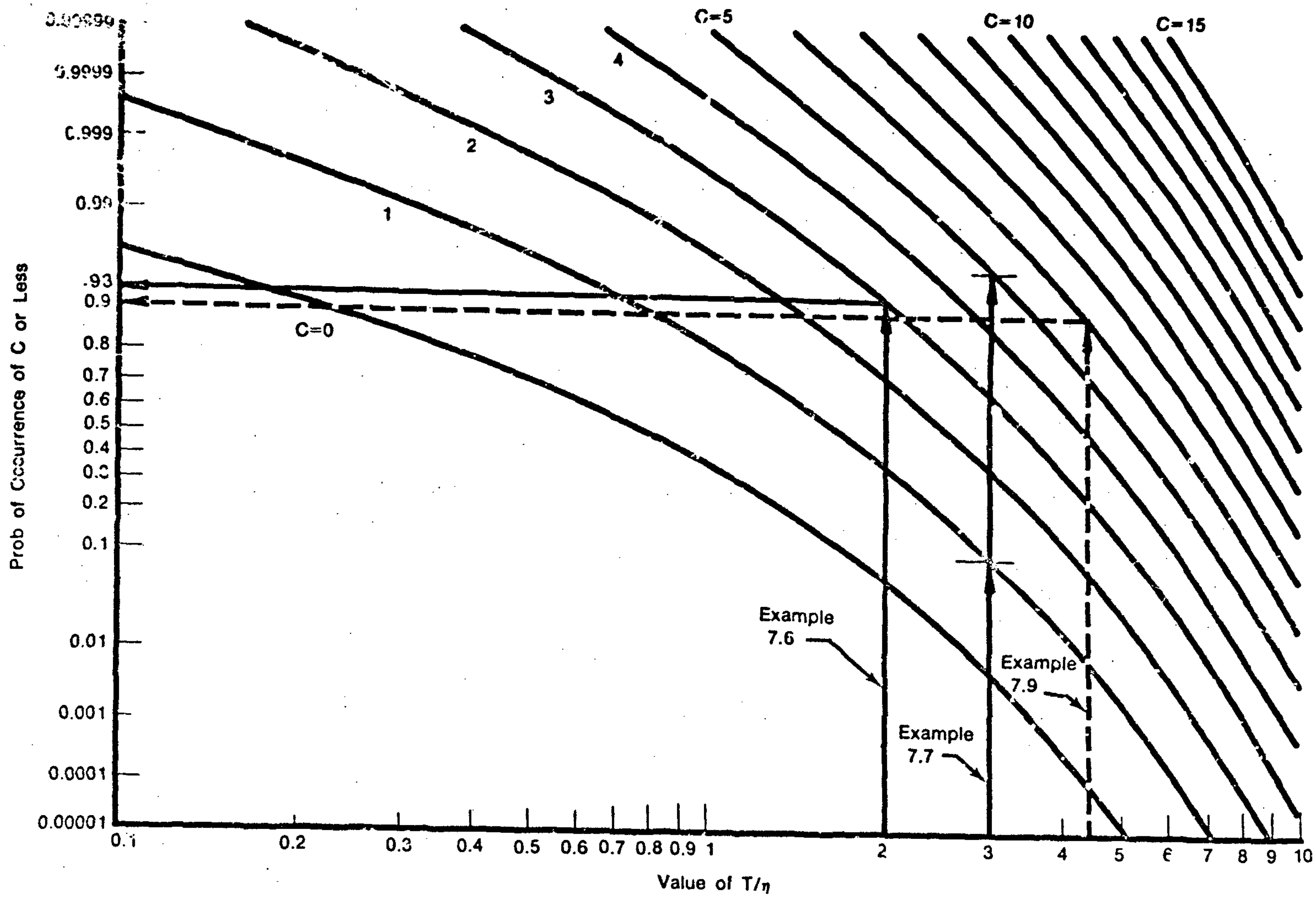
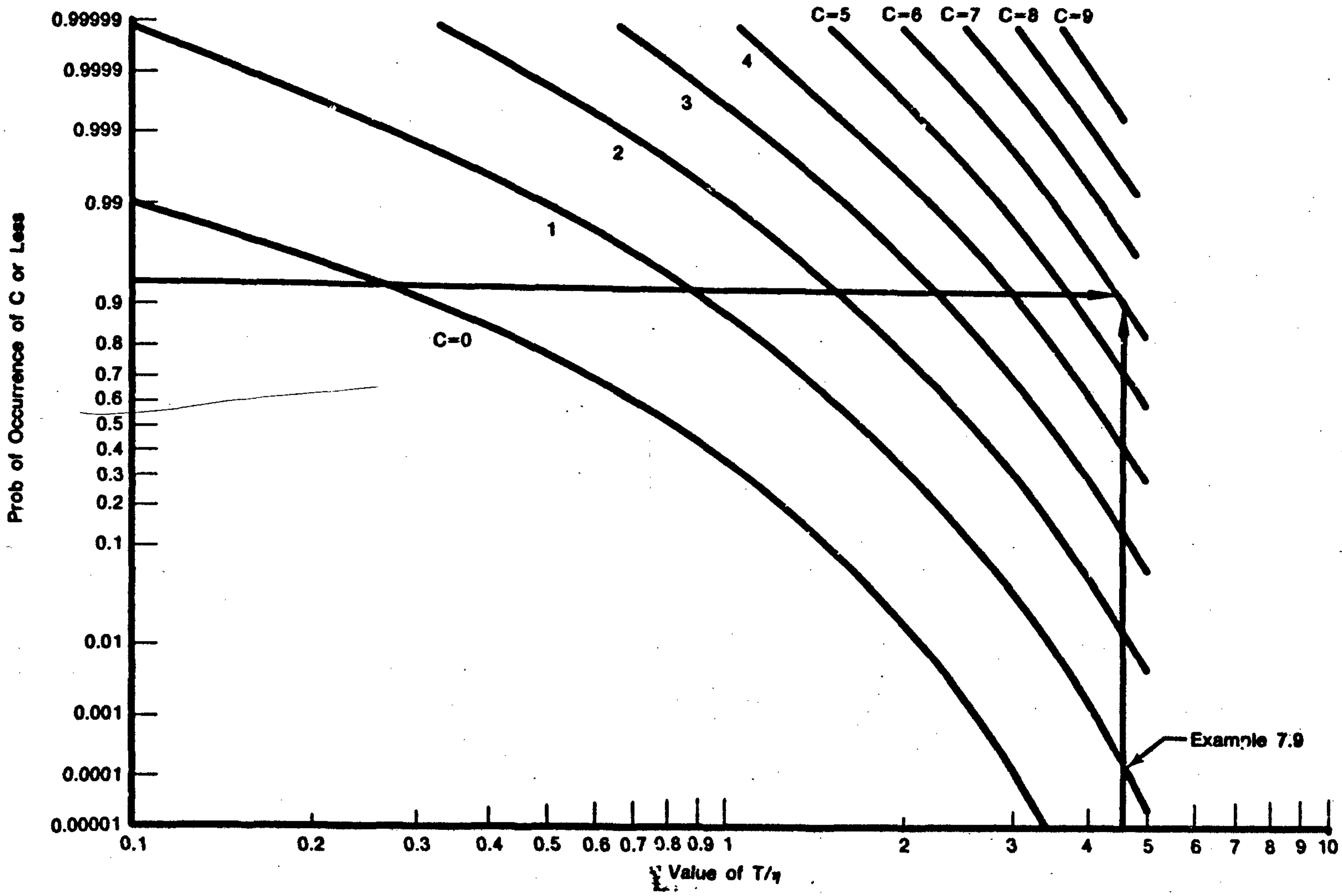


Figure 7.6. Weibull Thorndike Chart for β = 1.0



FD 256458

Figure 7.7. *Weibull Thorndike Chart for $\beta = 1.5$*



FD 256452

Figure 7.8. Weibull Thorndike Chart for $\beta = 2.0$

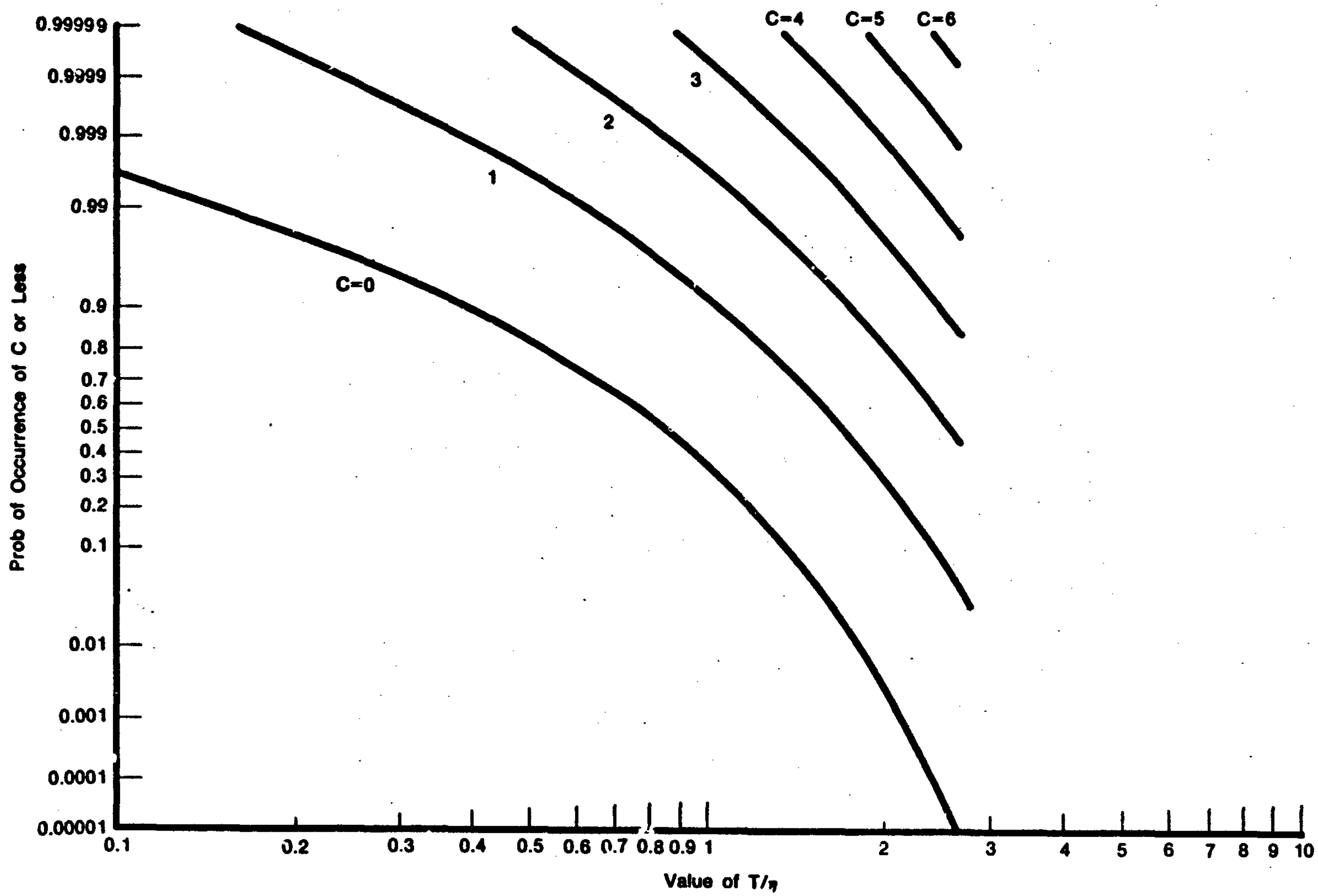
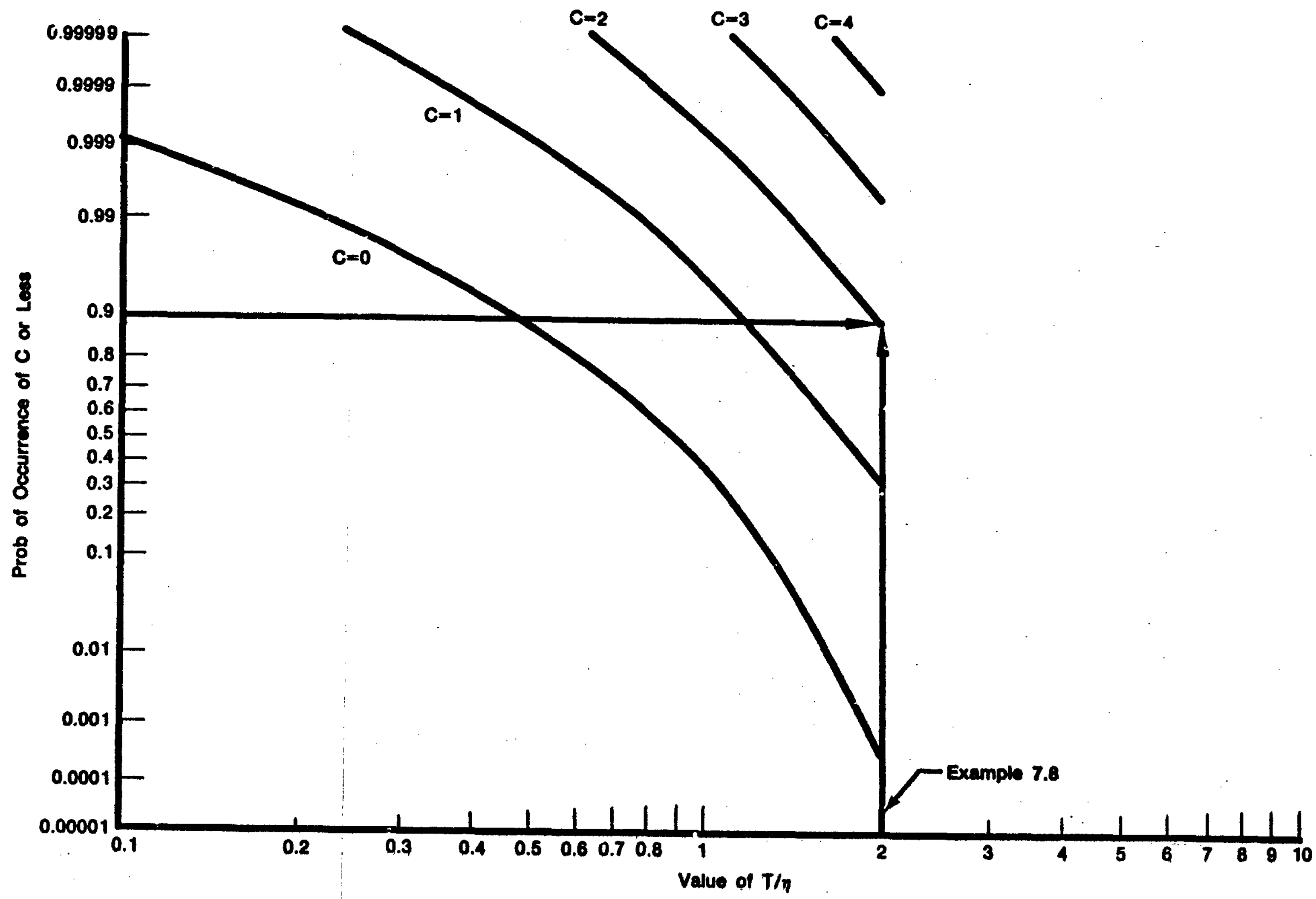
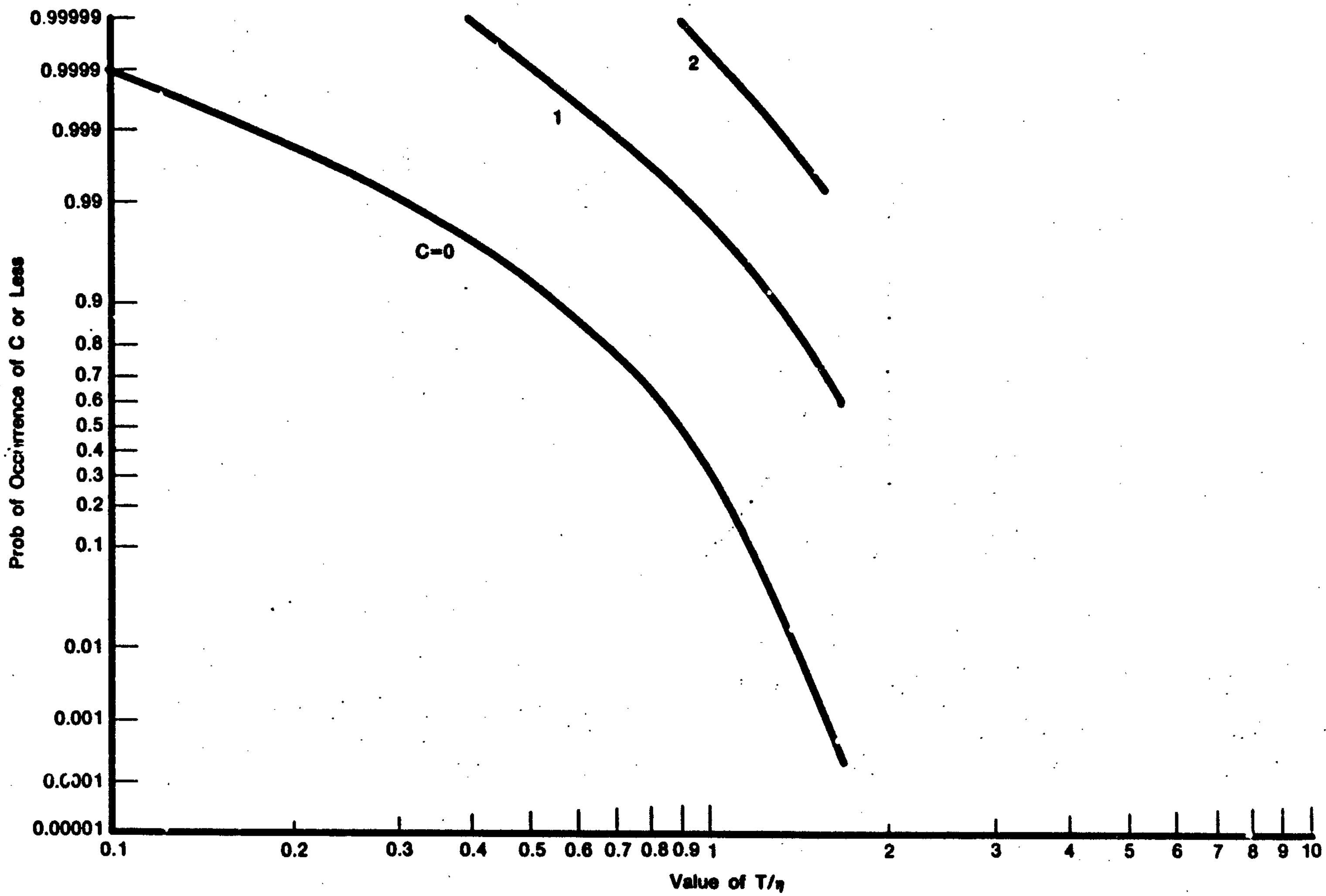


Figure 7.9. Weibull Thorndike Chart for β = 2.5



FD 256461

Figure 7.10. Weibull Thorndike Chart for $\beta = 3.0$



FD 256462

Figure 7.11. Weibull Thorndike Chart for β = 4.0

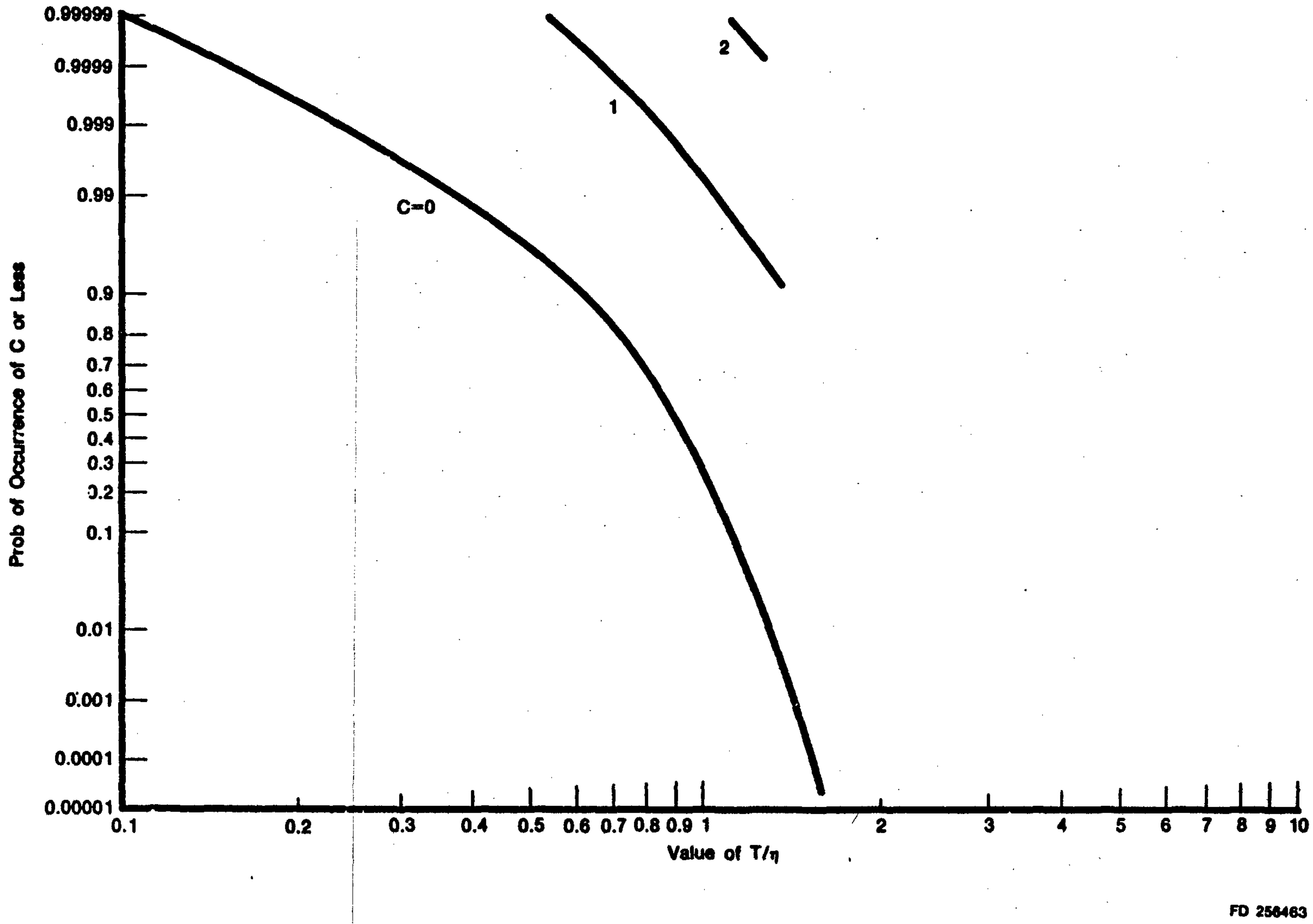


Figure 7.12. Weibull Thorndike Chart for $\beta = 5.0$

7.8 SHIFTING WEIBULLS

In all that has been done up to this point Weibull failure distributions have been estimated from observed failures, often in combination with the populations of unfailed units. What can be done if the times on each failure are known but the times on the population of unfailed units are unknown?

This problem arises in the failure analysis of data from jet engines because all parts in an engine are not serialized; that is, the time on the individual parts cannot be tracked. In many engines only the most important 400-500 parts are serialized, while the others (possibly as many as 10,000) are not.

Of course, if the part failure times are known, the engineer can generate a Weibull distribution from the failures only, using the methods in Chapter 2. Fred Dauser, Statistician, Commercial Products Division, Pratt & Whitney Aircraft Group, United Technologies Corporation, developed a method to "adjust" this Weibull if the number of unfailed units in the population is known.

An outline of the method is as follows:

1. Plot the failure data on Weibull probability paper.

2. Estimate the Weibull parameters β and η.

3. Calculate the mean time to failure (MTTF),

$$\text{MTTF} = \frac{\Sigma \text{ times to failure for each part}}{\text{No. failures}} \tag{7.5}$$

(now refer to Figure 7.13).

4. Draw a vertical line through the MTTF.

5. Calculate the proportion failed in the total population, No. failures/(No. failures + No. suspensions), calculate the cumulative % failed point = $(1 - e^{-\text{proportion}}) \times 100$, and draw a horizontal line from this point.

6. At the intersection of the vertical and horizontal lines draw a line parallel to the failure distribution. This is an estimate of the "true" Weibull distribution.

Example 7.10

Suppose there have been four flange failures with times of 1165, 1300, 1393, and 1493 cycles in a population of 2500; however, the times on the unfailed units are unknown. The procedure to estimate the "true" Weibull distribution can be used:

Steps 1 and 2: See Figure 7.14, $\beta = 9.53$, $\eta = 1400.7$ cycles

Step 3:

$$\text{MTTF} = \frac{1165 + 1300 + 1393 + 1493}{4} = 1337.8$$

No. failures/(No. failures + No. suspensions) = (4/2500) = 0.0016

Therefore, cum % failed = $[1 - e^{-0.0016}] \times 100 = 0.16\%$

Steps 4, 5 and 6:

See Figure 7.15

The estimated distribution has a β = 9.53 (same as the four failure Weibull), but the characteristic life is now = 2629 cycles.

7.9 WEIBULL GOODNESS OF FIT

The procedure to test whether a sample is from a specified Weibull distribution can be given in terms of the confidence bounds about the Weibull line developed in Section 7.6. Again, complete samples will be assumed.

Procedure:

1. Using the Weibull estimates of β and η for the failure distributions, calculate and plot the confidence bounds using the techniques of Section 7.6.

2. Now place the hypothesized Weibull on this same plot, as a dotted line.

3. If this dotted line does not lie entirely within the confidence bands, then consider the sample to be from a different Weibull distribution.

Example 7.11

Given a sample of seven failures with β = 2.0, η = 2000, as in Section 7.6, can the hypothesis that the sample comes from a Weibull distribution with β = 4.29 and η = 1500 be rejected?

Plotting the hypothesized Weibull as a dotted line on Figure 7.3 gives Figure 7.16. One would have to reject the hypothesis that this sample comes from a Weibull distribution with β = 4.29 and η = 1500 since the dotted line does not lie entirely within the bands.

7.10 COMPARING THE WEIBULL TO OTHER DISTRIBUTIONS

Figure 7.17 shows what happens if failure data from log-normal, normal, and extreme-value distributions are plotted on Weibull probability paper. Since the log-normal is the most frequent alternative in failure analysis, this will cover most of the practical cases that arise. For example, suppose two plots like those on Figure 7.18 are given. In this case, the eye is unable to discern which is "best," the log-normal or the Weibull. The statistical test from Reference[3] can be used to discriminate between these two failure distributions. The test can be set up in two ways: to favor the log-normal or to favor the Weibull.

[3] "Discrimination between the log-normal and Weibull Distributions," Dumonceaux and Antle, *Technometrics*, Vol. 15, 4, 1973.

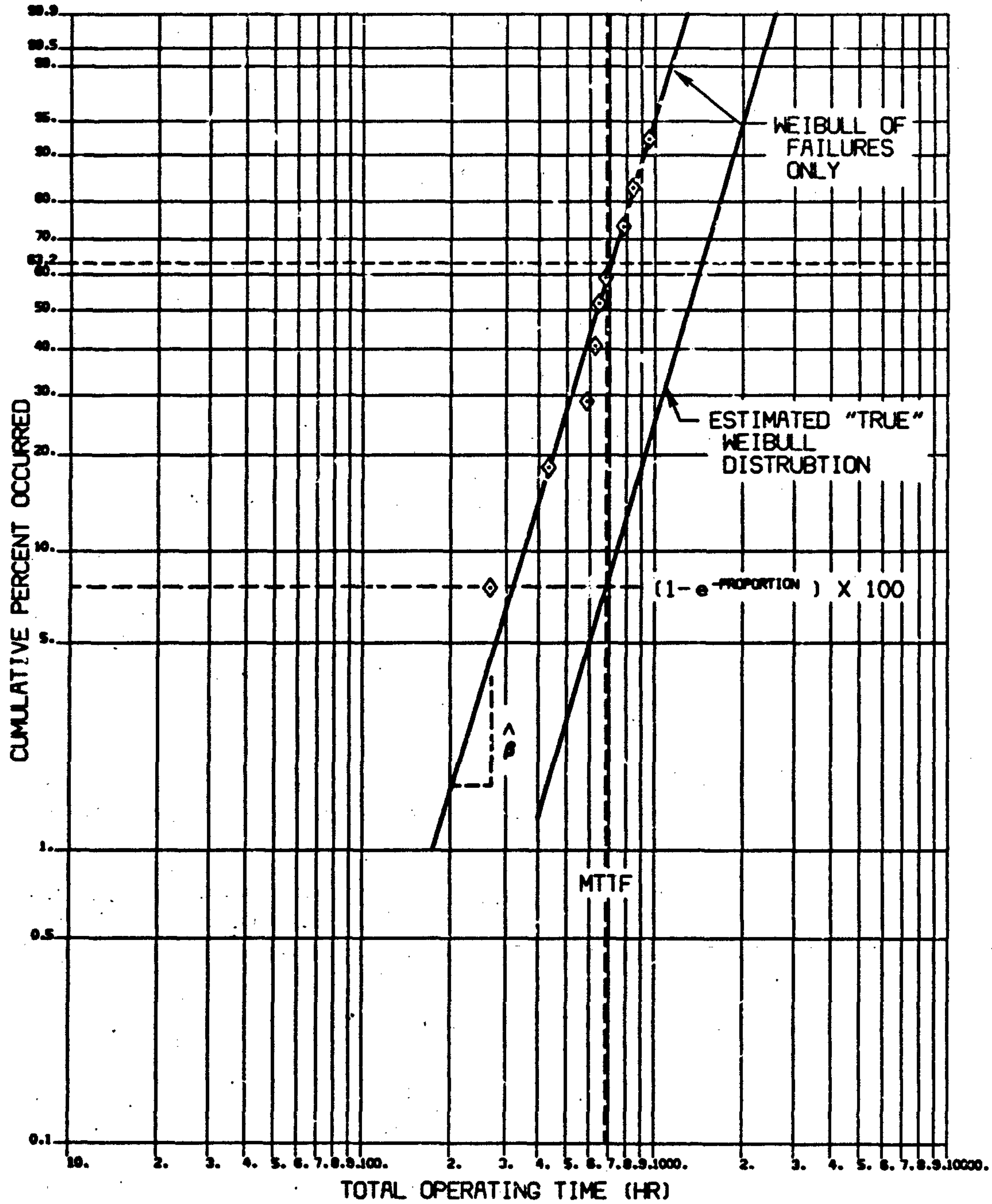


Figure 7.13. Example of Shifting a Weibull

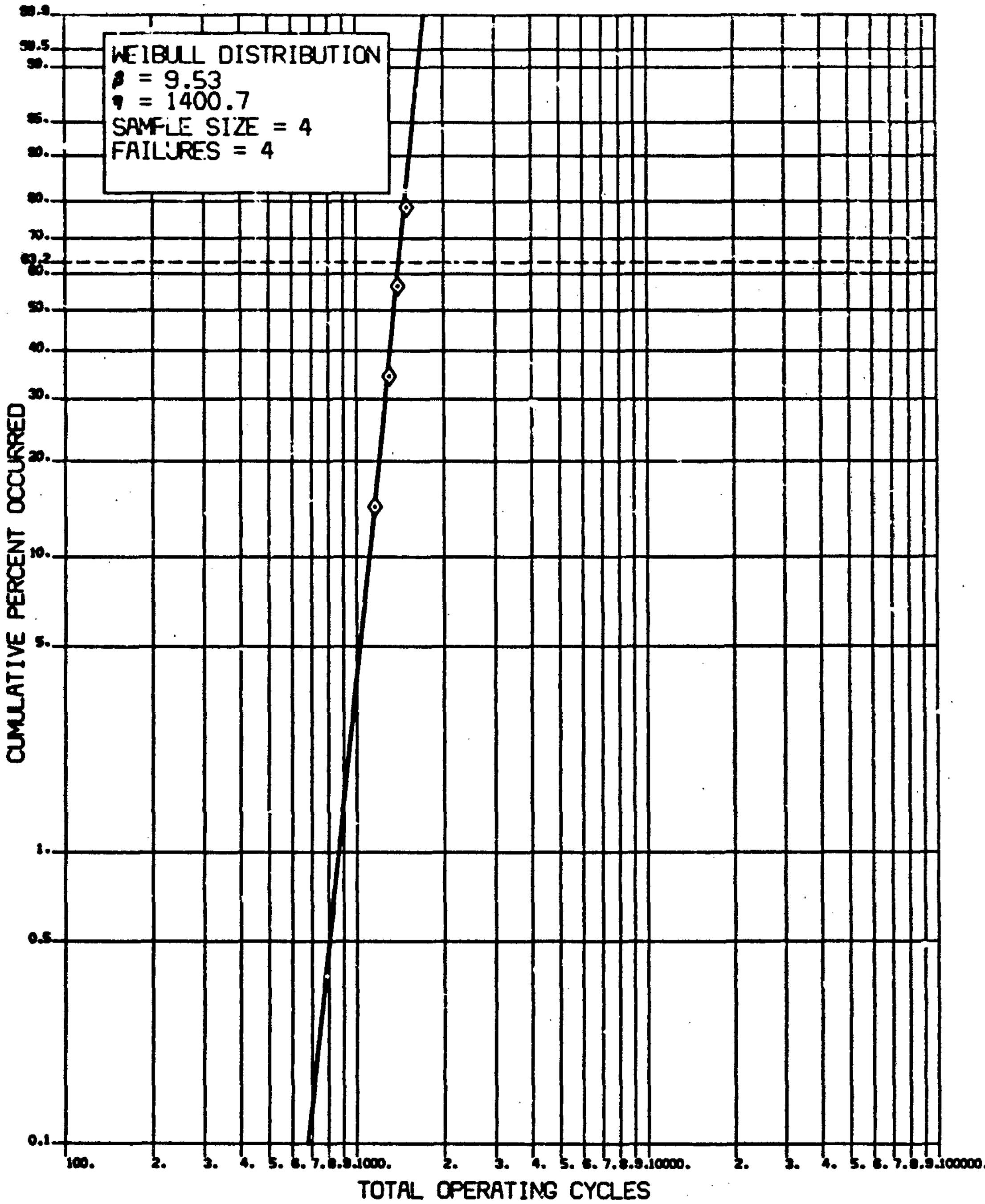


Figure 7.14. Flange Cracking

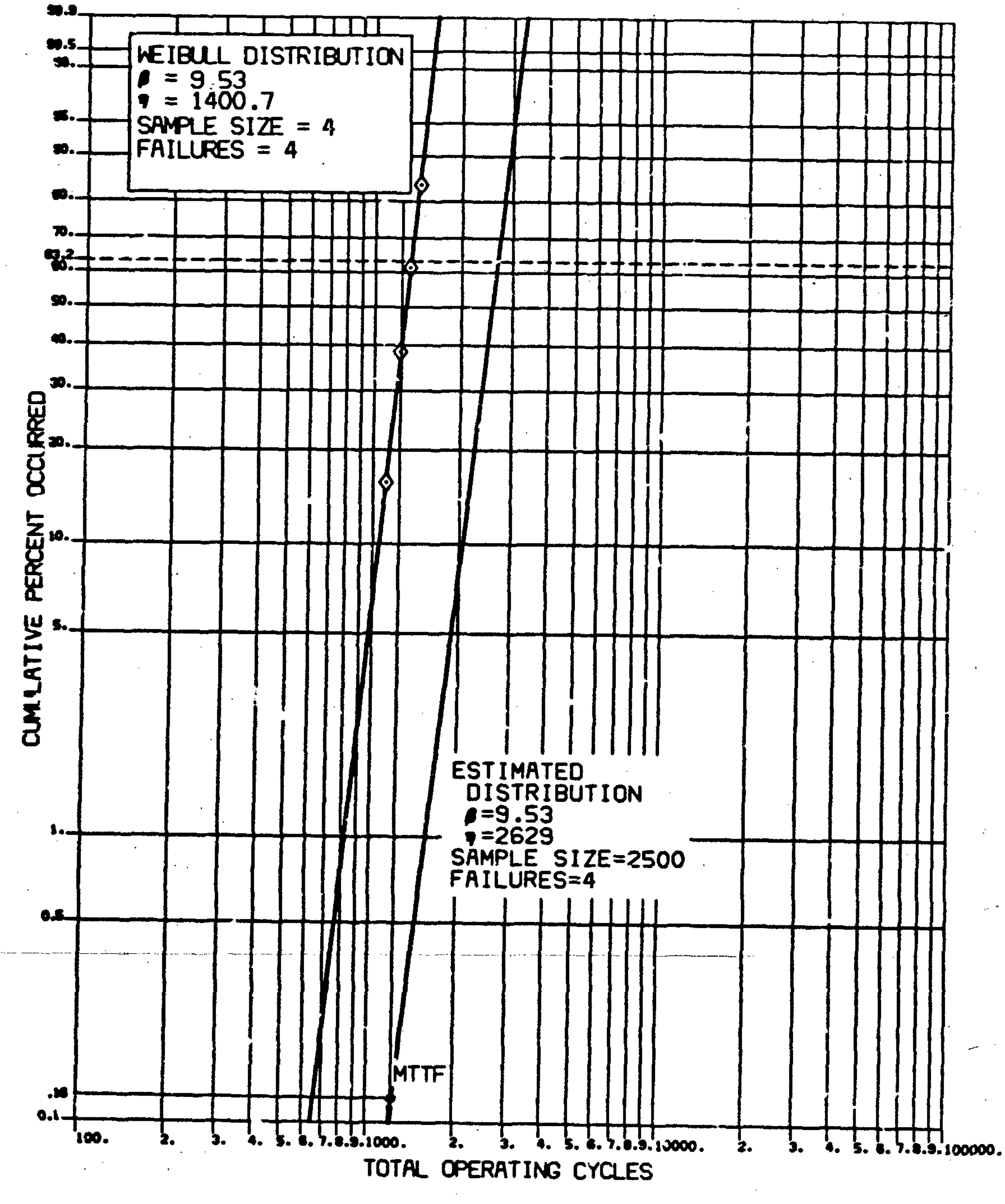


Figure 7.15. Flange Cracking Estimated Distribution

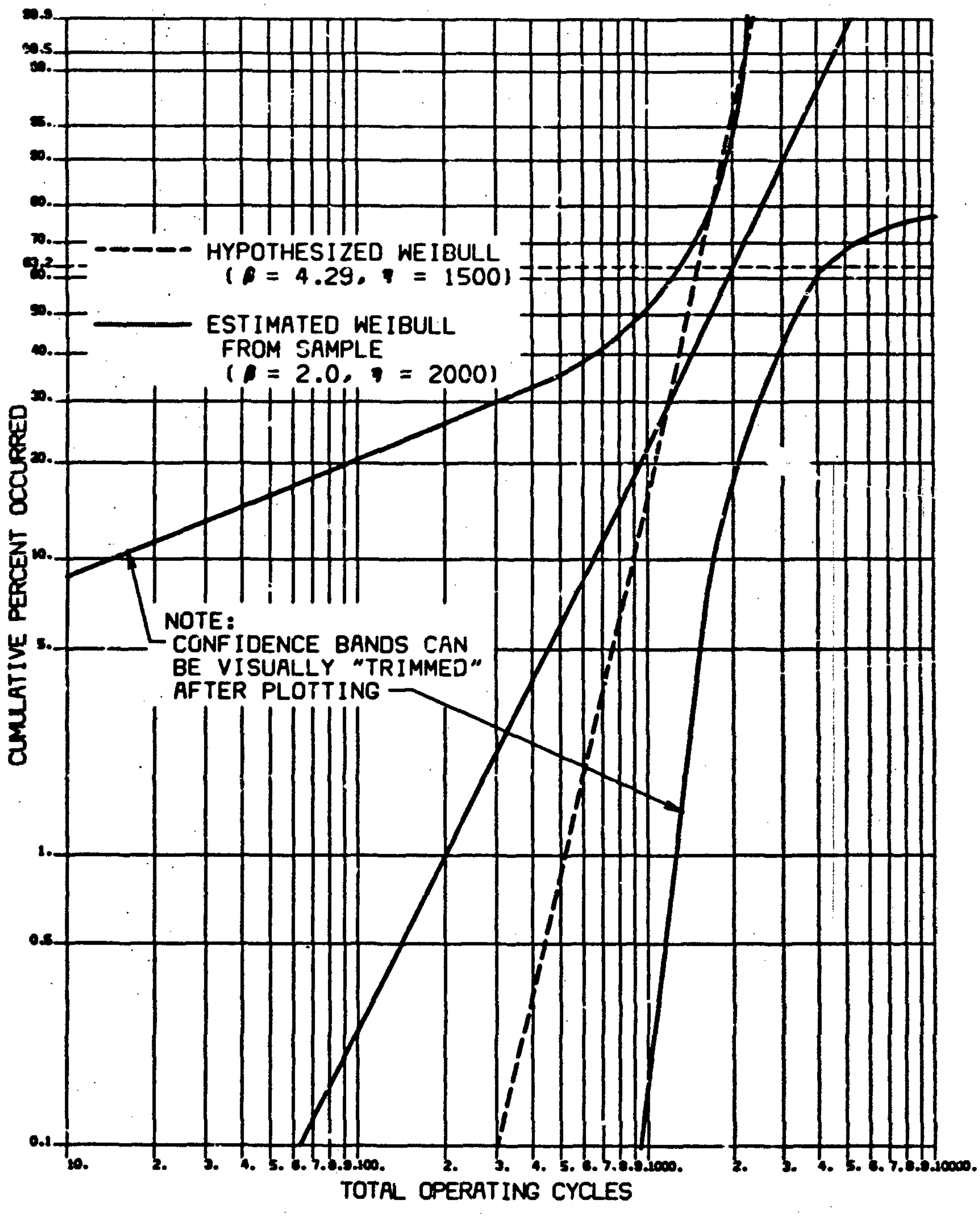


Figure 7.16. Hypothesized Weibull

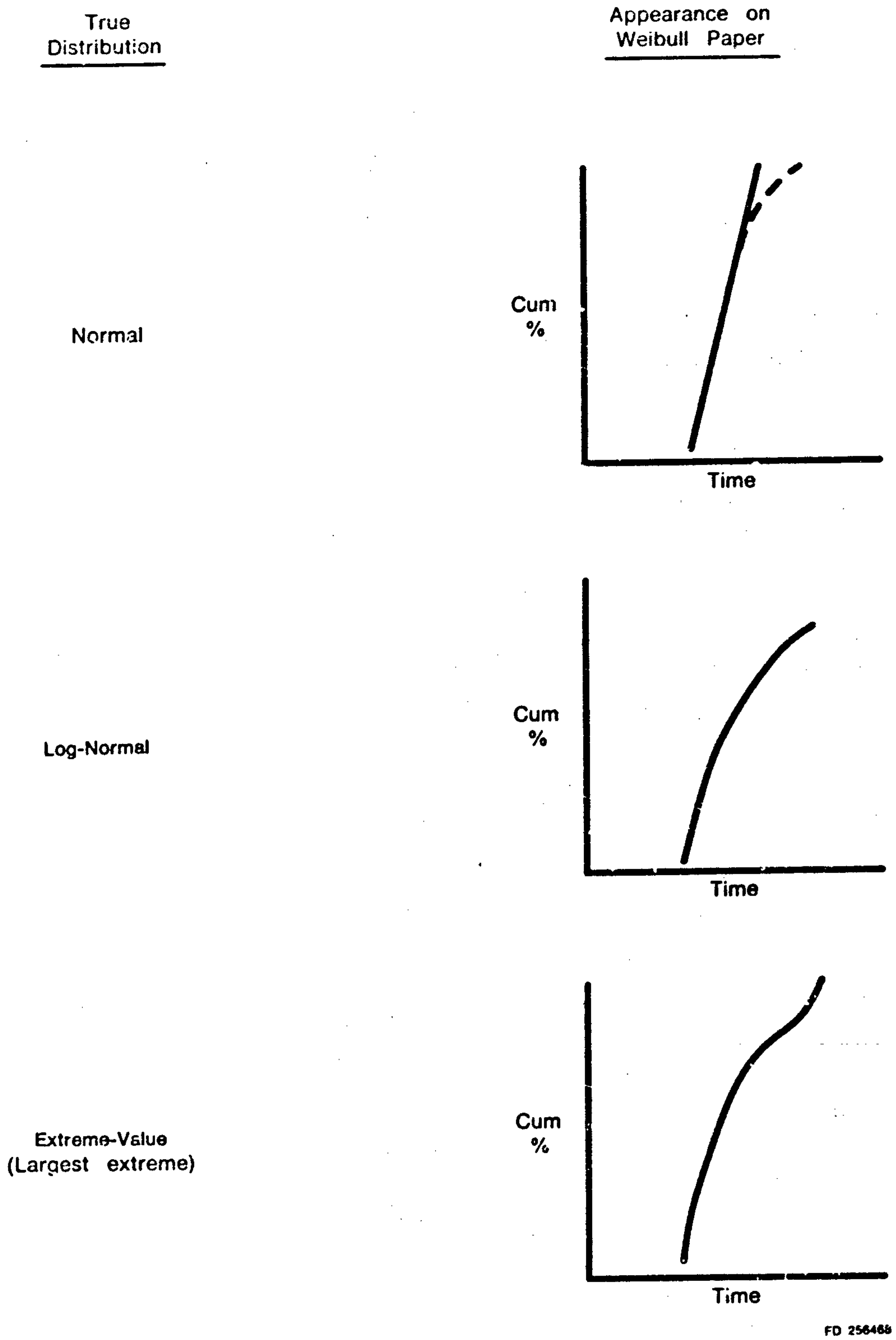


Figure 7.17. Picking the Best Distribution

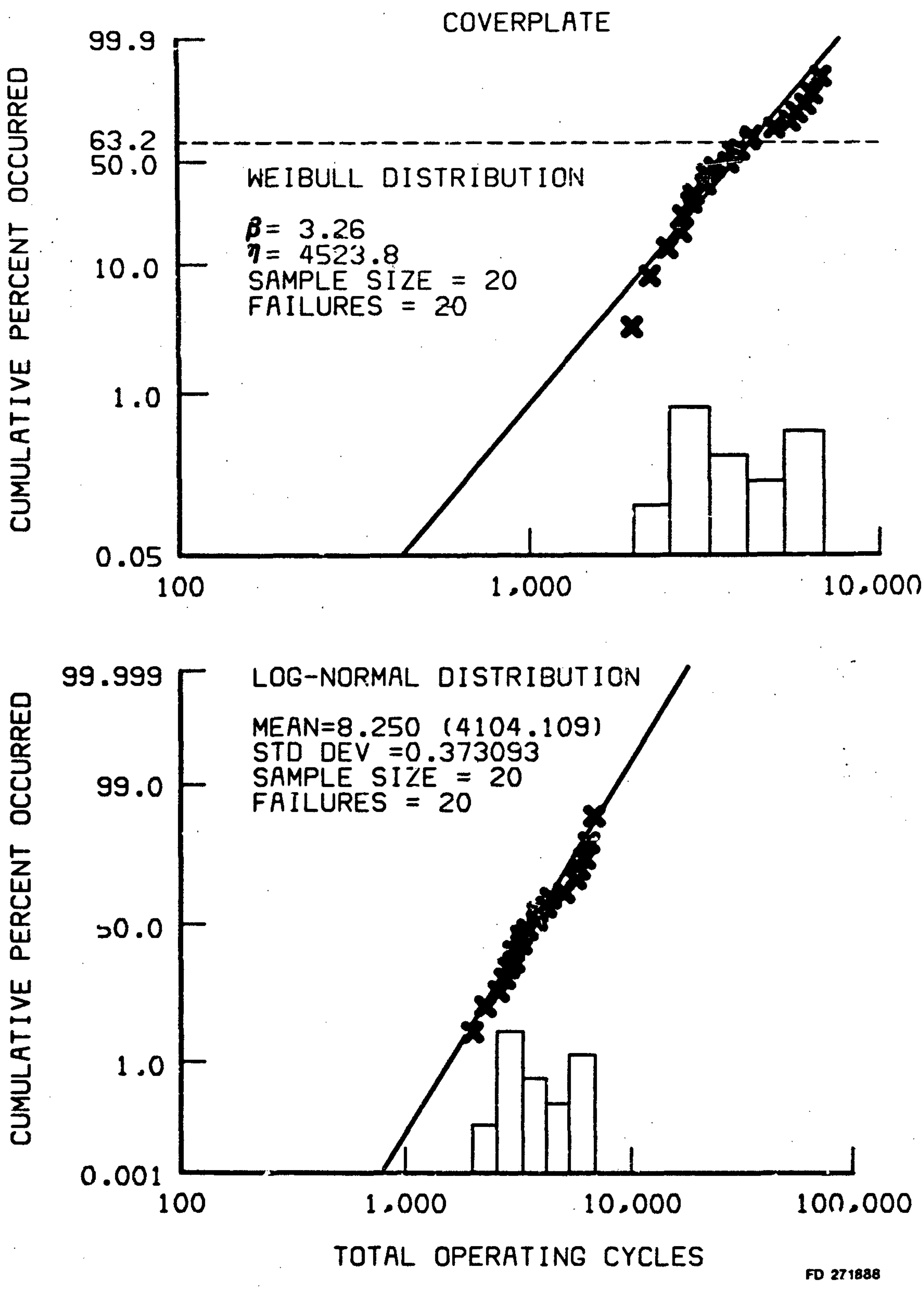


Figure 7.18. Comparing Weibull and Lognormal Distributions for Coverplate Failures

Favoring the Weibull, calculate

$$W = (2\pi e \hat{\sigma}^2)^{1/2} \quad (|t_1 f(t_1)|\,|t_2 f(t_2)| \ldots |t_n f(t_n)|)^{1/n} \tag{7.6}$$

$$\hat{\sigma}^2 = \frac{\Sigma(\ell n t_i - \hat{\mu})^2}{n}, \qquad n \text{ is sample size}$$

$$\hat{\mu} = \frac{1}{n}\Sigma \ell n t_i, \qquad t_i \text{ is the time of failure}$$

and $f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-(t/\eta)^\beta}$, the Weibull probability density function

W is compared to the appropriate table value for the confidence level desired (Table 7.3) and, if $W > W_{table}$, the Weibull is rejected in favor of the log-normal.

TABLE 7.3. CRITICAL VALUES FOR TESTING THE DIFFERENCE BETWEEN LOG NORMAL AND WEIBULL (FAVORING THE WEIBULL)

Number of Failures	*Confidence Level* 80%	90%	95%
20	1.008	1.041	1.067
30	0.991	1.019	1.041
40	0.980	1.005	1.026
50	0.974	0.995	1.016

Favoring the log-normal, calculate

$$W = (2\pi e \hat{\sigma}^2)^{1/2} \cdot (|t_1 f(t_1)|\,|t_2 f(t_2)| \ldots |t_n f(t_n)|)^{1/n} \tag{7.7}$$

and compare to the appropriate table value for the confidence level desired (Table 7.4) and, if $W \geq W_{table}$, the log-normal is rejected in favor of the Weibull.

TABLE 7.4. CRITICAL VALUES FOR TESTING THE DIFFERENCE BETWEEN LOG NORMAL AND WEIBULL (FAVORING THE LOG-NORMAL)

Number of Failures	*Confidence Level* 80%	90%	95%
20	1.015	1.038	1.062
30	0.993	1.020	1.044
40	0.984	1.007	1.028
50	0.976	0.998	1.014

Example 7.12

The coverplate failures that went into the plots in Figure 7.18 occurred at 1989, 2160, 2569, 2758, 2813, 2979, 3016, 3283, 3294, 3503, 3853, 3916, 4294, 4462, 5178, 5716, 5984, 6378, 6556, and 7000 cycles. The estimated Weibull parameters are $\hat{\beta} = 3.26$, $\hat{\eta} = 4523.8$. The estimated log-normal parameters are $\hat{\mu}$ 8.25014, $\hat{\sigma}$ 0.373093 Which failure distribution fits the data better with 80% confidence?

Both tests, favoring first the Weibull and then the log-normal, will be performed.

Favoring the Weibull, using equation (7.6),

$$W = [2\,(3.141592)\,(2.71828)\,0.373093^2]^{-1/2} \quad ([1989\, f(1989)] \ldots [7000\, f(7000)])^{-1/20}$$
$$= 1.039$$

compared to a table value of 1.008. Hence, reject the Weibull.

Favoring the log-normal, using equation (7.7),

$$W = [2\,(3.141592)\,(2.71828)\,0.373093^2]^{+1/2} \quad ([1989\, f(1989)] \ldots [700\, f(7000)])^{+1/20}$$
$$= 0.962$$

compared to a table value of 1.015. Therefore, since the value W in the test that favors the Weibull $\geq W_{table}$, the Weibull can be rejected in favor of the log-normal. This same decision is reached in the test favoring the log-normal; in this case, since $W \leq W_{table}$, the log-normal cannot be rejected in favor of the Weibull.

In conclusion, the log-normal failure distribution seems to describe this coverplate failure mode better than the Weibull distribution.

7.11 PROBLEMS

1. Given a Weibull derived from 40 data points with $\beta = 1.5$, $\eta = 2000$ hours; what are the 90% confidence intervals for β and η?

2. What is the 90% confidence interval for Reliability at 1500 hours in problem 1?

3. What are the 90% confidence intervals about the first three failures in problem 1?

4. Given a Weibull with parameters $\beta = 1$, $\eta = 1000$ hours, what is the 90% probability band on the number of failures to be expected by 4000 hours?

5. A 10 point Weibull of failures only was generated and is illustrated in Figure 7.19. These failures are of a non-serialized part with a total population size of 2000. Adjust this 10 failure Weibull for the entire sample size. Note: failure times are 51, 79, 116, 164, 197, 230, 232, 327, 414, and 451 hours.

6. Are the Weibulls in Figures 7.20 and 7.21 significantly different? Assume the Weibull in Figure 7.20 is true.

Solutions to these problems are in Appendix J.

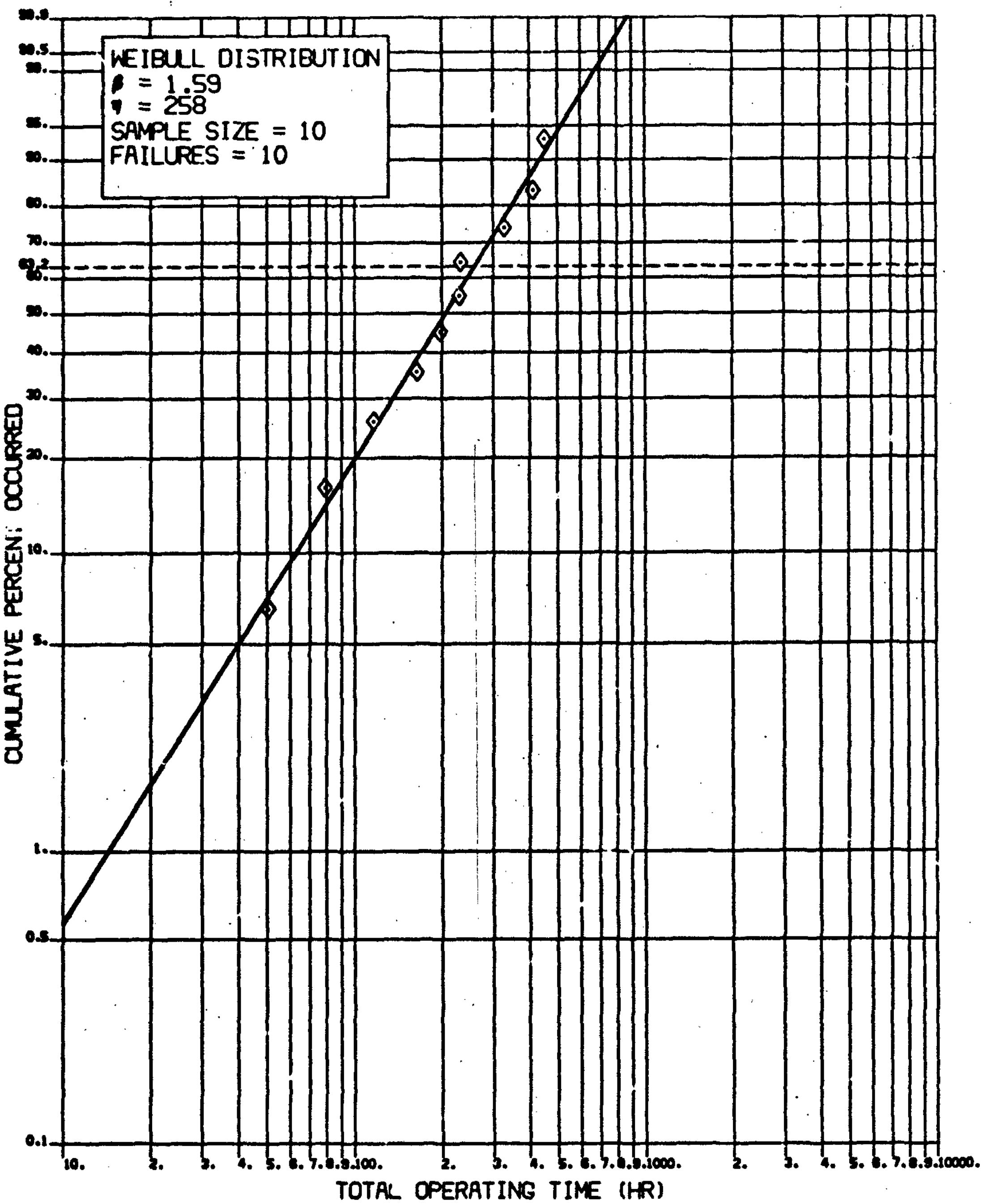


Figure 7.19. Weibull of Nonserialized Parts, Problem No. 5

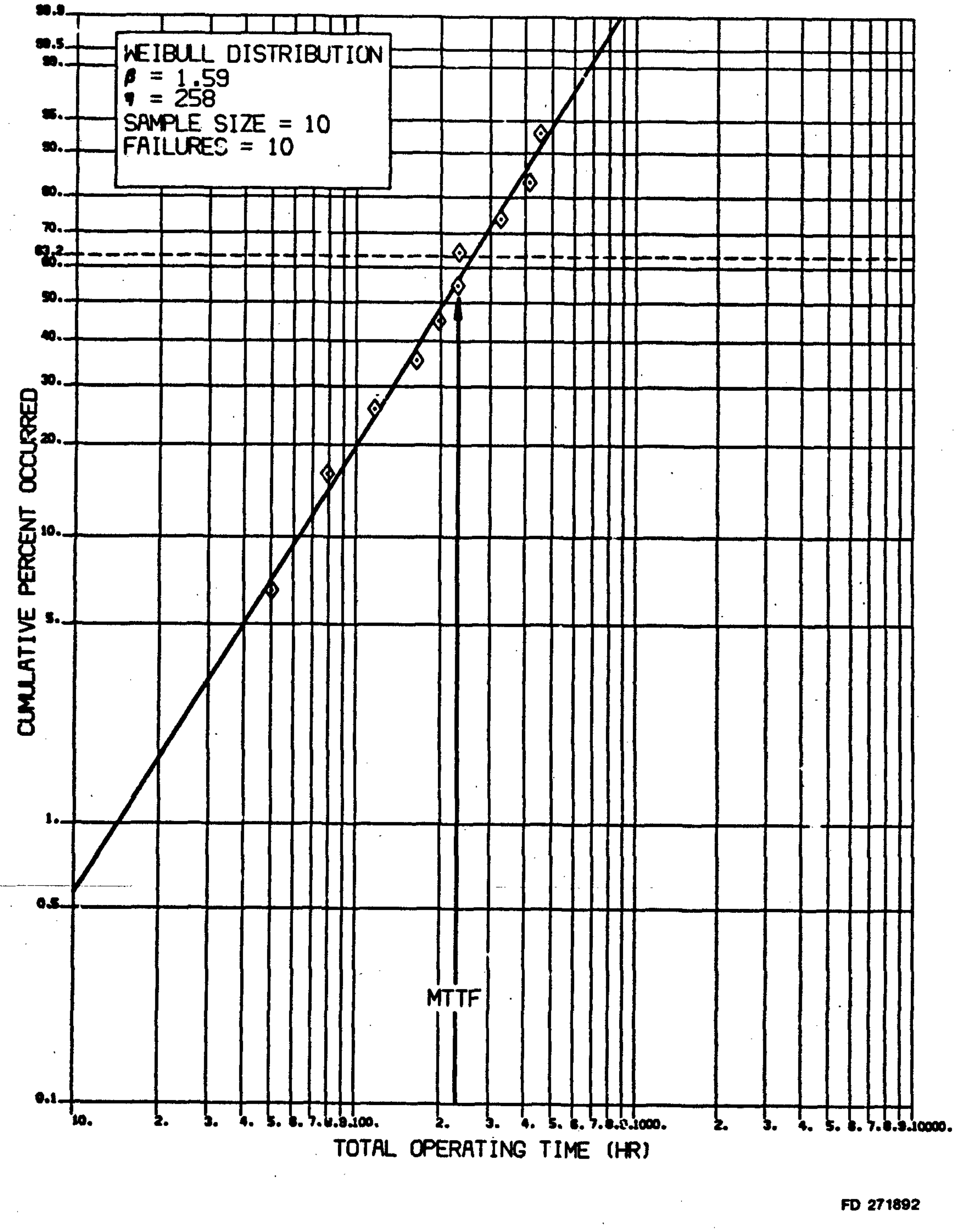
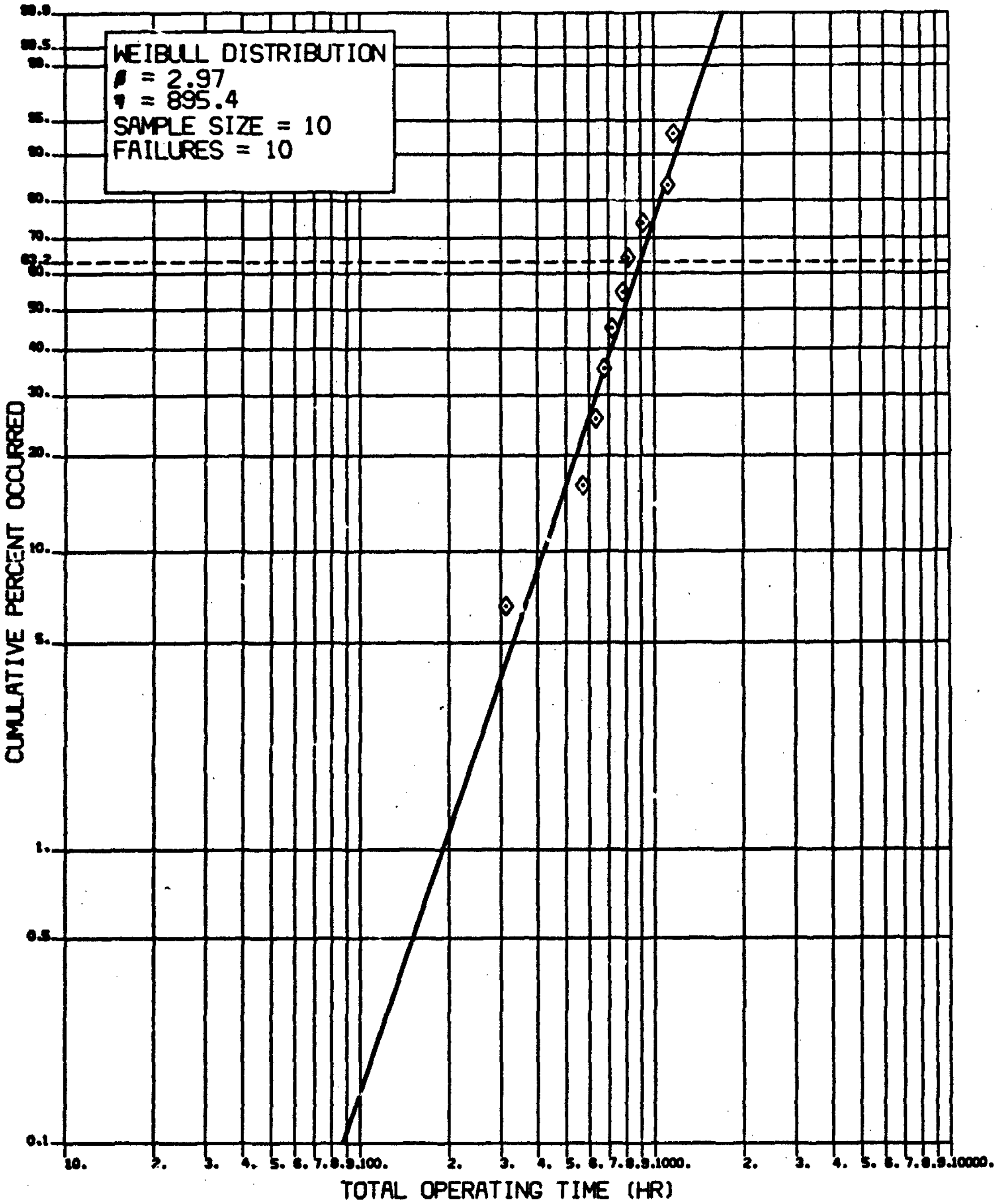


Figure 7.20. True Weibull, Problem No. 6



FD 271893

Figure 7.21. Suspect Failure Mode, Problem No. 6

APPENDIX A
GLOSSARY

1. β (Beta) — The parameter of the Weibull distribution that determines its shape and that implies the failure mode characteristic (infant mortality, random, or wearout). It is also called the slope parameter because it is estimated by the slope of the straight line on Weibull probability paper.

2. Bias — The difference between the true value of a population parameter and the grand average of many parameter estimates calculated from random samples drawn from the parent population. Also called fixed error.

3. Censored data — Data that contain suspended units.

4. Confidence — Relative frequency that the (statistically derived) interval contains the true value being estimated.

5. Distribution — A mathematical function giving the cumulative probability that a random quantity (e.g. a component's life) will be less than or equal to any given value.

6. η (Eta) — The characteristic life of the Weibull distribution. 63.2% of the lifetimes will be less than the characteristic life, regardless of the value of β, the Weibull slope parameter.

7. Hazard Rate — The instantaneous failure rate.

8. Infant Mortality — A failure mode characterized by a hazard rate that decreases with age, i.e., new units are more likely to fail than old units.

9. Monte Carlo Simulation — A mathematical model of a system with random elements, usually computer-adapted, whose outcome depends on the application of randomly generated numbers.

10. MTBF — Mean or average time between failures.

11. Parameter — An unknown constant associated with a population (such as the characteristic life of a Weibull population or the mean of a normal population).

12. Precision — The degree of agreement among estimates calculated from random samples drawn from a parent population. The precision is usually measured by the standard deviation of the estimates.

13. Random (failure mode) — A failure mode that is independent of time, in the sense that an old unit is as likely to fail as a new unit. In other words, the hazard rate remains constant with age.

14. Reliability — The probability that, when operating in the manner intended, a system will perform its intended function satisfactorily for a specified interval of time.

15. Risk Analysis — A prediction of the number of failures expected to occur in some future time period.

16. Suspension — A test or operational unit that has not failed by the mode under consideration at the time of the life data analysis.

17. t_o — Zero age for the failure mode. It is known as the minimum life parameter in the three-parameter Weibull distribution: units have zero probability of failure prior to t_o.

18. Wearout — A failure mode characterized by a hazard rate that increases with age, i.e., old units are more likely to fail than new units.

19. Weibayes/Weibest — A method for constructing a Weibull distribution based on assuming a value of β, the Weibull slope parameter. It is used when there are certain deficiencies in the data (for instance, when operating time has accumulated, but no failures have occurred).

20. Weibull Analysis — Procedure for finding the Weibull distribution that best describes a sample of unit lifetimes, in order to estimate reliability, determine failure mode characteristics, and predict the occurrences of future failures.

21. Weibull Plot — A plot of time-to-failure data on Weibull probability paper.

APPENDIX B

MEDIAN RANKS, 5% RANKS, AND 95% RANKS

TABLE B.1. MEDIAN RANKS

Rank Order	Sample Size *1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	50.0	29.2	20.6	15.9	12.9	10.9	9.4	8.3	7.4	6.6
2		70.7	50.0	38.5	31.3	26.4	22.8	20.1	17.9	16.2
3			79.3	61.4	50.0	42.1	36.4	32.0	28.6	25.8
4				84.0	68.6	57.8	50.0	44.0	39.3	35.5
5					87.0	73.5	63.5	55.9	50.0	45.1
6						89.0	77.1	67.9	60.6	54.8
7							90.5	79.8	71.3	64.4
8								91.7	82.0	74.1
9									92.5	83.7
10										93.3

Rank Order	Sample Size *11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
1	6.1	5.6	5.1	4.8	4.5	4.2	3.9	3.7	3.5	3.4
2	14.7	13.5	12.5	11.7	10.9	10.2	9.6	9.1	8.6	8.2
3	23.5	21.6	20.0	18.6	17.4	16.3	15.4	14.5	13.8	13.1
4	32.3	29.7	27.5	25.6	23.9	22.4	21.1	20.0	18.9	18.0
5	41.1	37.5	35.0	32.5	30.4	28.5	26.9	25.4	24.1	22.9
6	50.0	45.9	42.5	39.5	36.9	34.7	32.7	30.9	29.3	27.8
7	58.8	54.0	50.0	46.5	43.4	40.8	38.4	36.3	34.4	32.7
8	67.6	62.1	57.4	53.4	50.0	46.9	44.2	41.8	39.6	37.7
9	76.4	70.2	64.9	60.4	56.5	53.0	50.0	47.2	44.8	42.6
10	85.2	78.3	72.4	67.4	63.0	59.1	55.7	52.7	50.0	47.5
11	93.8	86.4	79.9	74.3	69.5	65.2	61.5	58.1	55.1	52.4
12		94.3	87.4	81.3	76.0	71.4	67.2	63.6	60.3	57.3
13			94.8	88.2	82.5	77.5	73.0	69.0	65.5	62.2
14				95.1	89.3	83.6	78.8	74.5	70.6	67.2
15					95.4	89.7	84.5	79.9	75.8	72.1
16						95.7	90.3	85.4	81.0	77.0
17							96.0	90.8	86.1	81.9
18								96.2	91.3	86.8
19									96.4	91.7
20										96.5

TABLE B.1. MEDIAN RANKS

Rank Order	Sample Size 21	22	23	24	25	26	27	28	29	30
1	3.2	3.1	2.9	2.8	2.7	2.6	2.5	2.4	2.3	2.2
2	7.8	7.5	7.1	6.8	6.6	6.3	6.1	5.9	5.7	5.5
3	12.5	11.9	11.4	10.9	10.5	10.1	9.7	9.4	9.1	8.8
4	17.2	16.4	15.7	15.0	14.4	13.9	13.4	12.9	12.5	12.1
5	21.8	20.9	20.0	19.1	18.4	17.7	17.0	16.4	15.9	15.3
6	26.5	25.3	24.2	23.2	22.3	21.5	20.7	20.0	19.3	18.6
7	30.2	29.8	28.5	27.4	26.3	25.3	24.3	23.5	22.7	21.9
8	35.9	34.3	32.8	31.5	30.2	29.1	28.0	27.0	26.1	25.2
9	40.8	38.8	37.1	35.8	34.2	32.9	31.7	30.5	29.5	28.5
10	45.3	43.2	41.4	39.7	38.1	36.7	35.3	34.1	32.9	31.8
11	50.0	47.7	45.7	43.8	42.1	40.5	39.0	37.6	36.3	35.1
12	54.6	52.2	50.0	47.9	46.0	44.3	42.6	41.1	39.7	38.4
13	59.3	56.7	54.2	52.0	50.0	48.1	46.3	44.7	43.1	41.7
14	64.0	61.1	58.5	56.1	53.9	51.8	50.0	48.2	46.5	45.0
15	68.7	65.8	62.8	60.2	57.8	55.6	53.6	51.7	50.0	48.3
16	73.4	70.1	67.1	64.3	61.8	59.4	57.3	55.2	53.4	51.6
17	78.1	74.6	71.4	68.4	65.7	63.2	60.9	58.8	56.8	54.9
18	82.7	79.0	75.7	72.5	69.7	67.0	64.6	62.3	60.2	58.2
19	87.4	83.5	79.9	76.7	73.6	70.8	68.2	65.8	63.6	61.5
20	92.1	88.0	84.2	80.8	77.6	74.6	71.9	69.4	67.0	64.8
21	96.7	92.4	88.5	84.9	81.5	78.4	75.6	72.9	70.4	68.1
22		96.8	92.8	89.0	85.5	82.2	79.2	76.4	73.8	71.4
23			97.0	93.1	89.4	86.0	82.9	79.9	77.2	74.7
24				97.1	93.3	89.8	86.5	83.5	80.6	78.0
25					97.2	93.6	90.2	87.0	84.0	81.3
26						97.3	93.8	90.5	87.4	84.6
27							97.4	94.0	90.8	87.8
28								97.5	94.2	91.1
29									97.6	94.4
30										97.7

TABLE B.1. MEDIAN RANKS

Rank Order	Sample Size 31	32	33	34	35	36	37	38	39	40
1	2.2	2.1	2.0	2.0	1.9	1.9	1.8	1.8	1.7	1.7
2	5.3	5.1	5.0	4.8	4.7	4.6	4.4	4.3	4.2	4.1
3	8.5	8.2	8.0	7.7	7.5	7.3	7.1	6.9	6.7	6.6
4	11.7	11.3	11.0	10.6	10.3	10.1	9.8	9.5	9.3	9.1
5	14.9	14.4	14.0	13.6	13.2	12.8	12.5	12.1	11.8	11.5
6	18.0	17.5	17.0	16.5	16.0	15.6	15.1	14.7	14.4	14.0
7	21.2	20.6	20.0	19.4	18.8	18.3	17.8	17.3	16.9	16.5
8	24.4	23.7	23.0	22.3	21.7	21.1	20.5	20.0	19.4	19.0
9	27.6	26.8	26.0	25.2	24.5	23.8	23.2	22.6	22.0	21.4
10	30.8	29.9	29.0	28.1	27.3	26.6	25.8	25.2	24.5	23.9
11	34.0	32.9	32.0	31.0	30.1	29.3	28.5	27.8	27.1	26.4
12	37.2	36.0	35.0	33.9	33.0	32.1	31.2	30.4	29.6	28.9
13	40.4	39.1	38.0	36.8	35.8	34.8	33.9	33.0	32.2	31.4
14	43.6	42.2	41.0	39.8	38.6	37.6	36.6	35.6	34.7	33.8
15	46.8	45.3	44.0	42.7	41.5	40.3	39.2	38.2	37.2	36.3
16	50.0	48.4	47.0	45.6	44.3	43.1	41.9	40.5	39.8	38.8
17	53.1	51.5	50.0	48.5	47.1	45.8	44.6	43.4	42.3	41.3
18	56.3	54.6	52.9	51.4	50.0	48.6	47.3	46.0	44.9	43.8
19	59.5	57.7	55.9	54.3	52.8	51.3	50.0	48.6	47.4	46.2
20	62.7	60.8	58.9	57.2	55.6	54.1	52.6	51.3	50.0	48.7
21	65.9	63.9	61.9	60.1	58.4	56.8	55.3	53.9	52.5	51.2
22	69.1	67.0	64.9	63.1	61.3	59.6	58.0	56.5	55.0	53.7
23	72.3	70.0	67.9	66.0	64.1	62.3	60.7	59.1	57.6	56.1
24	75.5	73.1	70.9	68.9	66.9	65.1	63.3	61.7	60.1	58.6
25	78.7	76.2	73.9	71.8	69.8	67.8	66.0	64.3	62.7	61.1
26	81.9	79.3	76.9	74.7	72.6	70.6	68.7	66.9	65.2	63.6
27	85.0	82.4	79.9	77.6	75.4	73.3	71.4	69.5	67.7	66.1
28	88.2	85.5	82.9	80.5	78.2	76.1	74.1	72.1	70.3	68.5
29	91.4	88.6	85.9	83.4	81.1	78.8	76.7	74.7	72.8	71.0
30	94.6	91.7	88.9	86.3	83.9	81.6	79.4	77.3	75.4	73.5
31	97.7	94.8	91.9	89.3	86.7	84.3	82.1	79.9	77.9	76.0
32		97.8	94.9	92.2	89.6	87.1	84.8	82.6	80.5	78.5
33			97.9	95.1	92.4	89.8	87.4	85.2	83.0	80.9
34				97.9	95.2	92.6	90.1	87.8	85.5	83.4
35					98.0	95.3	92.8	90.4	88.1	85.9
36						98.0	95.5	93.0	90.8	88.4
37							98.1	95.6	93.2	90.8
38								98.1	95.7	93.3
39									98.2	95.8
40										98.2

TABLE B.1. MEDIAN RANKS

Rank Order	Sample Size 41	42	43	44	45	46	47	48	49	50
1	1.6	1.6	1.5	1.5	1.5	1.4	1.4	1.4	1.4	1.3
2	4.0	3.9	3.8	3.7	3.7	3.6	3.5	3.4	3.4	3.3
3	6.4	6.3	6.1	6.0	5.8	5.7	5.6	5.5	5.4	5.3
4	8.8	8.6	8.4	8.2	8.0	7.9	7.7	7.5	7.4	7.2
5	11.3	11.0	10.7	10.5	10.3	10.0	9.8	9.6	9.4	9.2
6	13.7	13.3	13.0	12.7	12.5	12.2	11.9	11.7	11.4	11.2
7	16.1	15.7	15.3	15.0	14.7	14.3	14.0	13.7	13.5	13.2
8	18.5	18.1	17.6	17.2	16.9	16.5	16.2	15.8	15.5	15.2
9	20.9	20.4	20.0	19.5	19.1	18.7	18.3	17.9	17.5	17.2
10	23.3	22.8	22.3	21.8	21.3	20.8	20.4	20.0	19.5	19.2
11	25.8	25.2	24.6	24.0	23.5	23.0	22.5	22.0	21.6	21.1
12	28.2	27.5	26.9	26.3	25.7	25.1	24.6	24.1	23.6	23.1
13	30.6	29.9	29.2	28.5	27.9	27.3	26.7	26.2	25.6	25.1
14	33.0	32.2	31.5	30.8	30.1	29.4	28.8	28.2	27.7	27.1
15	35.4	34.6	33.8	33.0	32.3	31.6	30.9	30.3	29.7	29.1
16	37.9	37.0	36.1	35.3	34.5	33.8	33.1	32.4	31.7	31.1
17	40.3	39.3	38.4	37.5	36.7	35.9	35.2	34.4	33.7	33.1
18	42.7	41.7	40.7	39.8	38.9	38.1	37.3	36.5	35.8	35.1
19	45.1	44.0	43.0	42.1	41.1	40.2	39.4	38.6	37.8	37.0
20	47.5	46.4	45.3	44.3	43.3	42.4	41.5	40.6	39.8	39.0
21	50.0	48.8	47.6	46.6	45.5	44.6	43.6	42.7	41.8	41.0
22	52.4	51.1	50.0	48.8	47.7	46.7	45.7	44.8	43.9	43.0
23	54.8	53.5	52.3	51.1	50.0	48.9	47.8	46.8	45.9	45.0
24	57.2	55.9	54.6	53.3	52.2	51.0	50.0	48.9	47.9	47.0
25	59.6	58.2	56.9	55.6	54.4	53.2	52.1	51.0	50.0	49.0
26	62.0	60.6	59.2	57.8	56.6	55.3	54.2	53.1	52.0	50.9
27	64.5	62.9	61.5	60.1	58.8	57.5	56.3	55.1	54.0	52.9
28	66.9	65.3	63.8	62.4	61.0	59.7	58.4	57.2	56.0	54.9
29	69.3	67.7	66.1	64.6	63.2	61.8	60.5	59.3	58.1	56.9
30	71.7	70.0	68.4	66.9	65.4	64.0	62.6	61.3	60.1	58.9
31	74.1	72.4	70.7	69.1	67.6	66.1	64.7	63.4	62.1	60.9
32	76.6	74.8	73.0	71.4	69.8	68.3	66.9	65.5	64.1	62.9
33	79.0	77.1	75.3	73.6	72.0	70.5	69.0	67.5	66.2	64.8
34	81.4	79.5	77.6	75.9	74.2	72.6	71.1	69.6	68.2	66.8
35	83.8	81.8	79.9	78.1	76.4	74.8	73.2	71.7	70.2	68.8
36	86.2	84.2	82.3	80.4	78.6	76.9	75.3	73.7	72.2	70.8
37	88.7	86.6	84.6	82.7	80.8	79.1	77.4	75.8	74.3	72.8
38	91.1	88.9	86.9	84.9	83.0	81.2	79.5	77.9	76.3	74.8
39	93.5	91.3	89.2	87.2	85.2	83.4	81.6	79.9	78.3	76.8
40	95.9	93.6	91.5	89.4	87.4	85.6	83.7	82.0	80.4	78.8
41	98.3	96.0	93.8	91.7	89.6	87.7	85.9	84.1	82.4	80.7
42		98.3	96.1	93.9	91.9	89.9	88.0	86.2	84.4	82.7
43			98.4	96.2	94.1	92.0	90.1	88.2	86.4	84.7
44				98.4	96.2	94.2	92.2	90.3	88.5	86.7
45					98.4	96.3	94.3	92.4	90.5	88.7
46						98.5	96.4	94.4	92.5	90.7
47							98.5	96.5	94.5	92.7
48								98.5	96.5	94.6
49									98.5	96.6
50										98.6

TABLE B.2. FIVE PERCENT RANKS

Rank Order	Sample Size 1	2	3	4	5	6	7	8	9	10
1	5.0	2.5	1.6	1.2	1.0	0.8	0.7	0.6	0.5	0.5
2		22.3	13.5	9.7	7.8	6.2	5.3	4.6	4.1	3.6
3			36.8	24.8	18.9	15.3	12.8	11.1	9.7	8.7
4				47.2	34.2	27.1	22.5	19.2	16.8	15.0
5					54.9	41.8	34.1	28.9	25.1	22.2
6						60.6	47.9	40.0	34.4	30.3
7							65.1	52.9	45.0	39.3
8								68.7	57.0	49.3
9									71.6	60.5
10										74.1

Rank Order	Sample Size 11	12	13	14	15	16	17	18	19	20
1	0.4	0.4	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.2
2	3.3	3.0	2.8	2.6	2.4	2.2	2.1	2.0	1.9	1.8
3	7.8	7.1	6.6	6.1	5.6	5.3	4.9	4.7	4.4	4.2
4	13.5	12.2	11.2	10.4	9.6	9.0	8.4	7.9	7.5	7.1
5	19.9	18.1	16.5	15.2	14.1	13.2	12.3	11.6	10.9	10.4
6	27.1	24.5	22.3	20.6	19.0	17.7	16.6	15.6	14.7	13.9
7	34.9	31.5	28.7	26.3	24.3	22.6	21.1	19.8	18.7	17.7
8	43.5	39.0	35.4	32.5	29.9	27.8	26.0	24.3	22.9	21.7
9	52.9	47.2	42.7	39.0	35.9	33.3	31.0	29.1	27.3	25.8
10	63.5	56.1	50.5	45.9	42.2	39.1	36.4	34.0	32.0	30.1
11	76.1	66.1	58.9	53.4	48.9	45.1	41.9	39.2	36.8	34.6
12		77.9	68.8	61.4	56.0	51.5	47.8	44.9	41.8	39.3
13			79.4	70.3	63.6	58.3	53.9	50.2	47.0	44.1
14				80.7	72.0	65.6	60.4	56.1	52.4	49.2
15					81.8	73.6	67.3	62.3	58.0	54.4
16						82.9	74.9	68.9	64.0	59.8
17							83.8	76.2	70.4	65.6
18								84.6	77.3	71.7
19									85.4	78.3
20										86.0

TABLE B.2. FIVE PERCENT RANKS

Rank Order	Sample Size 21	22	23	24	25	26	27	28	29	30
1	0.2	0.2	0.2	0.2	0.2	0.1	0.1	0.1	0.1	0.1
2	1.7	1.6	1.5	1.5	1.4	1.3	1.3	1.2	1.2	1.1
3	4.0	3.8	3.6	3.4	3.3	3.2	3.0	2.9	2.8	2.7
4	6.7	6.4	6.1	5.9	5.6	5.4	5.2	5.0	4.8	4.6
5	9.8	9.4	8.9	8.5	8.2	7.8	7.5	7.3	7.0	6.8
6	13.2	12.6	12.0	11.4	11.0	10.5	10.1	9.7	9.4	9.0
7	16.8	15.9	15.2	14.5	13.9	13.3	12.8	12.3	11.9	11.4
8	20.5	19.5	18.6	17.7	17.0	16.3	15.6	15.0	14.5	14.0
9	24.4	23.2	22.1	21.1	20.2	19.3	18.6	17.9	17.2	16.6
10	28.5	27.1	25.8	24.6	23.5	22.5	21.6	20.8	20.0	19.3
11	32.8	31.1	29.6	28.2	26.9	25.8	24.7	23.8	22.9	22.1
12	37.1	35.2	33.5	31.9	30.5	29.2	28.0	26.9	25.8	24.9
13	41.7	39.5	37.5	35.7	34.1	32.6	31.3	30.0	28.9	27.8
14	46.4	43.9	41.6	39.6	37.6	36.2	34.6	33.3	32.0	30.8
15	51.2	48.4	45.9	43.7	41.6	39.8	38.1	36.6	35.2	33.8
16	56.3	53.1	50.3	47.8	45.6	43.5	41.7	40.0	38.4	36.9
17	61.5	58.0	54.9	52.1	49.6	47.3	45.3	43.4	41.7	40.1
18	67.0	63.0	59.6	56.5	53.7	51.3	49.0	47.0	45.1	43.3
19	72.9	68.4	64.5	61.0	58.0	55.3	52.8	50.6	48.5	46.6
20	79.3	74.0	69.6	65.8	62.4	59.4	56.7	54.2	52.0	50.0
21	86.7	80.1	75.0	70.7	67.0	63.7	60.7	58.1	55.7	53.4
22		87.2	80.9	76.0	71.8	68.1	64.9	62.0	59.4	57.0
23			87.7	81.7	76.8	72.8	69.2	66.0	63.2	60.6
24				88.2	82.3	77.7	73.7	70.2	67.1	64.2
25					88.7	83.0	78.4	74.5	71.1	68.1
26						89.1	83.6	79.1	75.3	72.0
27							89.4	84.1	79.8	76.1
28								89.8	84.6	80.4
29									90.1	85.1
30										90.4

TABLE B.2. FIVE PERCENT RANKS

Rank Order	Sample Size 31	32	33	34	35	36	37	38	39	40
1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
2	1.1	1.1	1.0	1.0	1.0	0.9	0.9	0.9	0.9	0.8
3	2.6	2.6	2.5	2.4	2.3	2.3	2.2	2.1	2.1	2.0
4	4.5	4.3	4.2	4.1	3.9	3.8	3.7	3.6	3.5	3.4
5	6.5	6.3	6.1	5.9	5.8	5.6	5.4	5.3	5.1	5.0
6	8.7	8.4	8.2	7.9	7.7	7.5	7.3	7.1	6.9	6.7
7	11.1	10.7	10.4	10.0	9.7	9.4	9.2	8.9	8.7	8.5
8	13.5	13.0	12.6	12.2	11.9	11.5	11.2	10.9	10.6	10.3
9	16.0	15.[illegible]	15.0	14.5	14.1	13.7	13.3	12.9	12.6	12.2
10	18.6	18.0	17.4	16.9	16.3	15.9	15.4	15.0	14.6	14.2
11	21.3	20.6	19.9	19.3	18.7	18.1	17.6	17.1	16.6	16.2
12	24.0	23.2	22.5	21.7	21.1	20.4	19.8	19.3	18.8	18.3
13	26.8	25.9	25.1	24.3	23.5	22.8	22.1	21.5	20.9	20.4
14	29.7	28.7	27.7	26.8	26.0	25.2	24.5	23.8	23.1	22.5
15	32.6	31.5	30.4	29.5	28.5	27.7	26.9	26.1	25.4	24.7
16	35.6	34.4	33.2	32.1	31.1	30.2	29.3	28.4	27.6	26.9
17	38.6	37.3	36.0	34.8	33.7	32.7	31.7	30.8	30.0	29.1
18	41.7	40.3	38.9	37.6	36.4	35.3	34.2	33.2	32.3	31.4
19	44.9	43.3	41.8	40.4	39.1	37.9	36.8	35.7	34.7	33.7
20	48.1	46.4	44.8	43.3	41.9	40.6	39.3	38.2	37.1	36.1
21	51.4	49.5	47.8	46.2	44.7	43.3	41.9	40.7	39.5	38.4
22	54.8	52.7	50.9	49.1	47.5	46.0	44.6	43.3	42.0	40.8
23	58.2	56.0	54.0	52.1	50.4	48.8	47.3	45.9	44.5	43.3
24	61.7	59.3	57.2	55.2	53.3	51.6	50.0	48.5	47.1	45.7
25	65.3	62.8	60.4	58.3	56.3	54.5	52.8	51.2	49.6	48.2
26	69.0	66.3	63.8	61.5	59.4	57.4	55.6	53.9	52.3	50.8
27	72.8	69.9	67.2	64.7	62.5	60.4	58.4	56.6	54.9	53.3
28	76.8	73.6	70.7	68.1	65.6	63.4	61.3	59.4	57.6	55.9
29	81.0	77.5	74.3	71.5	68.9	66.5	64.3	62.3	60.3	58.6
30	85.5	81.6	78.1	75.0	72.2	69.7	67.3	65.2	63.1	61.2
31	90.7	86.0	82.1	78.7	75.7	72.9	70.4	68.1	66.0	64.0
32		91.0	86.4	82.6	79.3	76.3	73.6	71.1	68.9	66.7
33			91.3	86.7	83.0	79.8	76.9	74.3	71.8	69.6
34				91.5	87.1	83.5	80.3	77.5	74.9	72.5
35					91.7	87.4	83.9	80.8	78.0	75.4
36						92.0	87.8	84.3	81.3	78.5
37							92.2	88.1	84.7	81.7
38								92.4	88.4	85.0
39									92.6	88.6
40										92.7

TABLE B.2. FIVE PERCENT RANKS

Rank Order	Sample Size 41	42	43	44	45	46	47	48	49	50
1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
2	0.8	0.8	0.8	0.8	0.7	0.7	0.7	0.7	0.7	0.7
3	2.0	1.9	1.9	1.8	1.8	1.8	1.7	1.7	1.6	1.6
4	3.4	3.3	3.2	3.1	3.0	3.0	2.9	2.8	2.8	2.7
5	4.9	4.8	4.6	4.5	4.4	4.3	4.2	4.1	4.1	4.0
6	6.5	6.4	6.2	6.1	5.9	5.9	5.7	5.5	5.4	5.3
7	8.2	8.0	7.8	7.7	7.5	7.3	7.2	7.0	6.9	6.7
8	10.0	9.5	9.6	9.3	9.1	8.9	8.7	8.5	8.3	8.2
9	11.9	11.6	11.3	11.1	10.8	10.6	10.3	10.1	9.9	9.7
10	13.8	13.5	13.1	12.8	12.5	12.2	12.0	11.7	11.5	11.2
11	15.8	15.4	15.0	14.6	14.3	14.0	13.7	13.4	13.1	12.8
12	17.8	17.3	16.9	16.5	16.1	15.7	15.4	15.1	14.7	14.4
13	19.8	19.3	18.9	18.4	18.0	17.5	17.2	16.8	16.4	16.1
14	21.9	21.4	20.8	20.3	19.8	19.4	18.9	18.5	18.1	17.7
15	24.0	23.4	22.8	22.3	21.7	21.2	20.8	20.3	19.9	19.4
16	26.2	25.5	24.9	24.3	23.7	23.1	22.6	22.1	21.6	21.2
17	28.4	27.6	26.9	26.3	25.6	25.0	24.5	23.9	23.4	22.9
18	30.6	29.8	29.0	28.3	27.6	27.0	26.4	25.8	25.2	24.7
19	32.8	32.0	31.2	30.4	29.6	28.9	28.3	27.6	27.0	26.5
20	35.1	34.2	33.3	32.5	31.7	30.9	30.2	29.5	28.9	28.3
21	37.4	36.4	35.5	34.6	33.7	32.9	32.2	31.5	30.8	30.1
22	39.7	38.7	37.7	36.7	35.8	35.0	34.2	33.4	32.6	31.9
23	42.1	41.0	39.9	38.9	37.9	37.0	36.2	35.3	34.5	33.8
24	44.5	43.3	42.2	41.1	40.1	39.1	38.2	37.3	36.5	35.7
25	46.9	45.6	44.4	43.3	42.2	41.2	40.2	39.3	38.4	37.6
26	49.3	48.0	46.7	45.5	44.4	43.3	42.3	41.3	40.4	39.5
27	51.8	50.4	49.1	47.8	46.6	45.5	44.4	43.3	42.4	41.4
28	54.3	52.8	51.4	50.1	48.8	47.6	46.5	45.4	44.4	43.4
29	56.9	55.3	53.8	52.4	51.1	49.8	48.6	47.5	46.4	45.3
30	59.5	57.8	56.2	54.8	53.4	52.0	50.8	49.6	48.4	47.3
31	62.1	60.3	58.7	57.1	55.7	54.3	52.9	51.7	50.5	49.3
32	64.8	62.9	61.2	59.5	58.0	56.5	55.1	53.8	52.6	51.4
33	67.5	65.5	63.7	62.0	60.4	58.8	57.4	56.0	54.7	53.4
34	70.3	68.2	66.3	64.5	62.7	61.1	59.6	58.2	56.8	55.5
35	73.1	70.9	68.9	67.0	65.2	63.5	61.9	60.4	58.9	57.6
36	76.0	73.7	71.5	69.5	67.6	65.9	64.2	62.6	61.1	59.7
37	79.0	76.5	74.3	72.1	70.2	68.3	66.5	64.9	63.3	61.8
38	82.1	79.5	77.0	74.8	72.7	70.8	68.9	67.2	65.5	64.0
39	85.4	82.5	79.9	77.5	75.3	73.3	71.3	69.5	67.8	66.2
40	88.9	85.7	82.9	80.3	78.0	75.8	73.8	71.9	70.1	68.4
41	92.9	89.1	86.0	83.3	80.8	78.4	76.3	74.3	72.4	70.6
42		93.1	89.4	86.8	83.6	81.1	78.9	76.8	74.8	72.9
43			93.2	89.6	86.6	83.9	81.5	79.3	77.2	75.3
44				93.4	89.8	86.9	84.3	81.9	79.7	77.6
45					93.5	90.0	87.2	84.6	82.2	80.1
46						93.6	90.3	87.4	84.9	82.6
47							93.8	90.4	87.7	85.2
48								93.9	90.6	87.9
49									94.0	90.8
50										94.1

TABLE B.3. NINETY-FIVE PERCENT RANKS

Rank Order	Sample Size 1	2	3	4	5	6	7	8	9	10
1	95.0	77.6	63.1	52.7	45.0	39.3	34.8	31.2	28.3	25.8
2		97.4	86.4	75.1	65.7	58.1	52.0	47.0	42.9	39.4
3			98.3	90.2	81.0	72.8	65.8	59.9	54.9	50.6
4				98.7	92.3	84.6	77.4	71.0	65.5	60.6
5					98.9	93.7	87.1	80.7	74.8	69.6
6						99.1	94.6	88.8	83.1	77.7
7							99.2	95.3	90.2	84.9
8								99.3	95.8	91.2
9									99.4	96.3
10										99.4

Rank Order	Sample Size 11	12	13	14	15	16	17	18	19	20
1	23.8	22.0	20.5	19.2	18.1	17.0	16.1	15.3	14.5	13.9
2	36.4	33.8	31.6	29.6	27.9	26.3	25.0	23.7	22.6	21.6
3	47.0	43.8	41.0	38.5	36.3	34.3	32.6	31.0	29.5	28.2
4	56.4	52.7	49.4	46.5	43.9	41.6	39.5	37.6	35.9	34.3
5	65.0	60.9	57.2	54.0	51.0	48.4	46.0	43.8	41.9	40.1
6	72.8	68.4	64.5	60.9	57.7	54.8	52.1	49.7	47.5	45.5
7	80.0	75.4	71.2	67.4	64.0	60.8	58.0	55.4	52.9	50.7
8	86.4	81.8	77.6	73.6	70.0	66.6	63.5	60.7	58.1	55.8
9	92.1	87.7	83.4	79.3	75.6	72.1	68.9	65.9	63.1	60.6
10	96.6	92.8	88.7	84.7	80.9	77.3	73.9	70.8	67.9	65.3
11	99.5	96.9	93.3	89.5	85.8	82.2	78.8	75.6	72.6	69.8
12		99.5	97.1	93.8	90.3	86.7	83.3	80.1	77.0	74.1
13			99.6	97.4	94.3	90.9	87.6	84.3	81.2	78.2
14				99.6	97.5	94.6	91.5	88.3	85.2	82.2
15					99.6	97.7	95.0	92.0	89.0	86.0
16						99.6	97.8	95.2	92.4	89.5
17							99.6	97.9	95.5	92.8
18								99.7	98.0	95.7
19									99.7	98.1
20										99.7

3471C

TABLE B.3. NINETY-FIVE PERCENT RANKS

Rank Order	Sample Size 21	22	23	24	25	26	27	28	29	30
1	13.2	12.7	12.2	11.7	11.2	10.8	10.5	10.1	9.8	9.5
2	20.6	19.8	19.0	18.2	17.6	16.9	16.3	15.8	15.3	14.8
3	27.0	25.9	24.9	23.9	23.1	22.2	21.5	20.8	20.1	19.5
4	32.9	31.5	30.3	29.2	28.1	27.1	26.2	25.4	24.6	23.8
5	38.4	36.9	35.4	34.1	32.9	31.8	30.7	29.7	28.8	27.9
6	43.6	41.9	40.3	38.9	37.5	36.2	35.0	33.9	32.8	31.8
7	48.7	46.8	45.0	43.4	41.9	40.5	39.2	37.9	36.8	35.7
8	53.5	51.5	49.6	47.8	46.2	44.6	43.2	41.8	40.5	39.3
9	58.2	56.0	54.0	52.1	50.3	48.7	47.1	45.6	44.2	42.9
10	62.8	60.4	58.3	56.2	54.3	52.6	50.9	49.3	47.9	46.5
11	67.1	64.7	62.4	60.3	58.3	56.4	54.6	52.9	51.4	49.9
12	71.4	68.8	66.4	64.2	62.1	60.1	58.2	56.5	54.8	53.3
13	75.5	72.8	70.3	68.0	65.8	63.7	61.8	59.9	58.2	56.6
14	79.4	76.7	74.1	71.7	69.4	67.3	65.3	63.3	61.5	59.8
15	83.1	80.4	77.8	75.3	73.0	70.7	68.6	66.6	64.7	63.0
16	86.7	84.0	81.3	78.8	76.4	74.1	71.9	69.9	67.9	66.1
17	90.1	87.3	84.7	82.2	79.7	77.4	75.2	73.0	71.0	69.1
18	93.2	90.5	87.9	85.4	82.9	80.6	78.3	76.1	74.1	72.[illegible]
19	95.9	93.5	91.0	88.5	86.0	83.6	81.3	79.1	77.0	75.0
20	98.2	96.1	93.8	91.4	88.9	86.6	84.3	82.0	79.9	77.8
21	99.7	98.3	96.3	94.0	91.7	89.4	87.1	84.9	82.7	80.6
22		99.7	98.4	96.5	94.3	92.1	89.8	87.6	85.4	83.3
23			99.7	98.4	96.6	94.5	92.4	90.2	88.0	85.9
24				99.7	98.5	96.7	94.7	92.6	90.5	88.5
25					99.7	98.6	96.9	94.9	92.9	90.9
26						99.8	98.6	97.0	95.1	93.1
27							99.8	98.7	97.1	95.3
28								99.8	98.7	97.2
29									99.8	98.8
30										99.8

TABLE B.3. NINETY-FIVE PERCENT RANKS

Rank Order	Sample Size 31	32	33	34	35	36	37	38	39	40
1	9.2	8.9	8.6	8.4	8.2	7.9	7.7	7.5	7.3	7.2
2	14.4	13.9	13.5	13.2	12.8	12.5	12.1	11.8	11.5	11.3
3	18.9	18.3	17.8	17.3	16.9	16.4	16.0	15.6	15.2	14.9
4	23.1	22.4	21.8	21.2	20.6	20.1	19.6	19.1	18.6	18.2
5	27.1	26.3	25.6	24.9	24.2	23.6	23.0	22.4	21.9	21.4
6	30.9	30.0	29.2	28.4	27.7	27.0	26.3	25.6	25.0	24.5
7	34.6	33.6	32.7	31.8	31.0	30.2	29.5	28.8	28.1	27.4
8	38.2	37.1	36.1	35.2	34.3	33.4	32.6	31.8	31.0	30.3
9	41.7	40.6	39.5	38.4	37.4	36.5	35.6	34.7	33.9	33.2
10	45.1	43.9	42.7	41.6	40.5	39.5	38.6	37.6	36.8	35.9
11	48.5	47.2	45.9	44.7	43.6	42.5	41.5	40.5	39.6	38.7
12	51.8	50.4	49.0	47.8	46.6	45.4	44.3	43.3	42.3	41.3
13	55.0	53.5	52.1	50.8	49.5	48.3	47.1	46.0	45.0	44.0
14	58.2	56.6	55.1	53.7	52.4	51.1	49.9	48.7	47.6	46.6
15	61.3	59.6	58.1	56.6	55.2	53.9	52.6	51.4	50.3	49.1
16	64.3	62.6	61.0	59.5	58.0	56.6	55.3	54.0	52.8	51.7
17	67.3	65.5	63.9	62.3	60.8	59.3	58.0	56.6	55.4	54.2
18	70.2	68.4	66.7	65.1	63.5	62.0	60.6	59.2	57.9	56.6
19	73.1	71.2	69.5	67.8	66.2	64.6	63.1	61.7	60.4	59.1
20	75.9	74.0	72.2	70.4	68.8	67.2	65.7	64.2	62.8	61.5
21	78.6	76.7	74.8	73.1	71.4	69.7	68.2	66.7	65.2	63.8
22	81.3	79.5	77.4	75.6	73.9	72.2	70.6	69.1	67.6	66.2
23	83.9	81.9	80.0	78.2	76.4	74.7	73.0	71.5	69.9	68.5
24	86.4	84.4	82.5	80.6	78.8	77.1	75.4	73.8	72.3	70.8
25	88.8	86.9	84.9	83.0	81.2	79.5	77.8	76.1	74.5	73.0
26	91.2	89.2	87.3	85.4	83.6	81.8	80.1	78.4	76.8	75.2
27	93.4	91.5	89.5	87.7	85.8	84.0	82.3	80.6	79.0	77.4
28	95.4	93.6	91.7	89.9	88.0	86.2	84.5	82.8	81.1	79.5
29	97.3	95.6	93.8	92.0	90.2	88.4	86.6	84.9	83.3	81.6
30	98.8	97.3	95.7	94.0	92.2	90.5	88.7	87.0	85.3	83.7
31	99.8	98.8	97.4	95.8	94.1	92.4	90.7	89.0	87.3	85.7
32		99.8	98.9	97.5	96.0	94.3	92.6	91.0	89.3	87.7
33			99.8	98.9	97.6	96.1	94.5	92.8	91.2	89.6
34				99.8	98.9	97.6	96.2	94.6	93.0	91.4
35					99.8	99.0	97.7	96.3	94.8	93.2
36						99.8	99.0	97.8	96.4	94.9
37							99.8	99.0	97.8	96.5
38								99.8	99.0	97.9
39									99.8	99.1
40										99.8

TABLE B.3. NINETY-FIVE PERCENT RANKS

Rank Order	Sample Size 41	42	43	44	45	46	47	48	49	50
1	7.0	6.8	6.7	6.5	6.4	6.3	6.1	6.0	5.9	5.8
2	11.0	10.8	10.5	10.3	10.1	9.9	9.7	9.5	9.3	9.1
3	14.5	14.2	13.9	13.6	13.3	13.0	12.7	12.5	12.2	12.0
4	17.8	17.4	17.0	16.6	16.3	16.0	15.6	15.3	15.0	14.7
5	20.9	20.4	20.0	19.6	19.1	18.8	18.4	18.0	17.7	17.3
6	23.9	23.4	22.9	22.4	21.9	21.5	21.0	20.6	20.2	19.8
7	26.8	26.2	25.6	25.1	24.6	24.1	23.6	23.1	22.7	22.3
8	29.6	29.0	28.4	27.8	27.2	26.6	26.1	25.6	25.1	24.6
9	32.4	31.7	31.0	30.4	29.7	29.1	28.6	28.0	27.5	27.0
10	35.1	34.4	33.6	32.9	32.3	31.6	31.0	30.4	29.8	29.3
11	37.8	37.0	36.2	35.4	34.7	34.0	33.4	32.7	32.1	31.5
12	40.4	39.6	38.7	37.9	37.2	36.4	35.7	35.0	34.4	33.7
13	43.0	42.1	41.2	40.4	39.5	38.8	38.0	37.3	36.6	35.9
14	45.6	44.6	43.7	42.8	41.9	41.1	40.3	39.5	38.8	38.1
15	48.1	47.1	46.1	45.1	44.2	43.4	42.5	41.7	41.0	40.2
16	50.6	49.5	48.5	47.5	46.5	45.6	44.8	43.9	43.1	42.3
17	53.0	51.9	50.8	49.8	48.8	47.9	47.0	46.1	45.2	44.4
18	55.4	54.3	53.2	52.1	51.1	50.1	49.1	48.2	47.3	46.5
19	57.8	56.6	55.5	54.4	53.3	52.3	51.3	50.3	49.4	48.5
20	60.2	58.9	57.7	56.6	55.5	54.4	53.4	52.4	51.5	50.6
21	62.5	61.2	60.0	58.8	57.7	56.6	55.5	54.5	53.5	52.6
22	64.8	63.5	62.2	61.0	59.8	58.7	57.6	56.6	55.5	54.6
23	67.1	65.7	64.4	63.2	62.0	60.8	59.7	58.6	57.5	56.5
24	69.3	67.9	66.6	65.3	64.1	62.9	61.7	60.6	59.5	58.5
25	71.5	70.1	68.8	67.4	66.2	64.9	63.7	62.6	61.5	60.4
26	73.7	72.3	70.9	69.5	68.2	67.0	65.7	64.6	63.4	62.3
27	75.9	74.4	73.0	71.6	70.3	69.0	67.7	66.5	65.4	64.2
28	78.0	76.5	75.0	73.6	72.3	71.0	69.7	68.5	67.3	66.1
29	80.1	78.5	77.1	75.6	74.3	72.9	71.6	70.4	69.1	68.0
30	82.1	80.6	79.1	77.6	76.2	74.9	73.5	72.3	71.0	69.8
31	84.1	82.6	81.0	79.6	78.2	76.8	75.4	74.1	72.9	71.6
32	86.1	84.5	83.0	81.5	80.1	78.7	77.3	76.0	74.7	73.4
33	88.0	86.4	84.9	83.4	81.9	80.5	79.1	77.8	76.5	75.2
34	89.9	88.3	86.8	85.3	83.8	82.4	81.0	79.6	78.3	77.0
35	91.7	90.1	88.6	87.1	85.6	84.2	82.7	81.4	80.0	78.7
36	93.4	91.9	90.3	88.8	87.4	85.9	84.5	83.1	81.8	80.5
37	95.0	93.5	92.1	90.6	89.1	87.7	86.2	84.8	83.5	82.2
38	96.5	95.1	93.7	92.2	90.8	89.3	87.9	86.5	85.2	83.8
39	97.9	96.6	95.3	93.8	92.4	91.0	89.6	88.2	86.8	85.5
40	99.1	98.0	96.7	95.4	94.0	92.6	91.2	89.8	88.4	87.1
41	99.8	99.1	98.0	96.8	95.5	94.1	92.7	91.4	90.0	88.7
42		99.8	99.1	98.1	96.9	95.6	94.2	92.9	91.6	90.2
43			99.8	99.1	98.1	96.9	95.7	94.4	93.0	91.7
44				99.8	99.2	98.1	97.0	95.8	94.5	93.2
45					99.8	99.2	98.2	97.1	95.3	94.6
46						99.8	99.2	98.2	97.1	95.9
47							99.8	99.2	98.3	97.2
48								99.8	99.2	98.3
49									99.8	99.2
50										99.8

APPENDIX C

RANK REGRESSION (WEIBULL PLOT) METHOD OF WEIBULL ANALYSIS

1. METHOD

Median rank regression uses a best-fit straight line, through the data plotted on Weibull paper, to estimate the Weibull parameters beta and eta. The "best-fit" line is found by the method of least squares.

First, the failure times and median ranks (see Chapter 2 for the calculation of median ranks) are "transformed", as follows:

$$Y = \ln(\text{cycles-to-failure})$$

$$X = \ln\left(\ln\left(\frac{1}{1 - \text{Median Rank of } Y}\right)\right)$$

(The median rank is expressed in decimal form.)

Least squares is then used to estimate A and B in the equation Y = A + BX. These estimates will be referred to as $\hat{A}$ and $\hat{B}$, respectively. The median rank regression estimates of the Weibull parameters are:

$$\hat{\beta} = \frac{1}{\hat{B}}$$

$$\hat{\eta} = e^{\hat{A}}$$

2. EXAMPLE AND STEP-BY-STEP PROCEDURE

The median rank regression method will be illustrated with the censored data listed below.

Cycles	*Status*
1500	Failure
1750	Suspension
2250	Failure
4000	Failure
4300	Failure
5000	Suspension
7000	Failure

Step 1: Calculate the median ranks of the failure times using the methods of Chapter 2.

Cycles to Failure	*Rank Order Number*	*Median Rank (decimal form)*
1500	1.0000	0.0946
2250	2.1667	0.2523
4000	3.3333	0.4099
4300	4.5000	0.5676
7000	6.2500	0.8041

Step 2: For each failure, calculate the natural (base e) logarithm of the cycles-to-failure ($Y = \ln$ (cycles-to-failure))

$$\text{and } X = \ln\left(\ln\left(\frac{1}{1 - \text{Median Rank of Y}}\right)\right)$$

Cycles to Failure	*Median Rank*	*Y*	*X*
1500	0.0946	7.3132	−2.3088
2250	0.2523	7.7187	−1.2353
4000	0.4099	8.2940	−0.6397
4300	0.5676	8.3664	−0.1763
7000	0.8041	8.8537	0.4887

Step 3: Calculate the least squares estimates $\hat{A}$ and $\hat{B}$ of A and B in the equation $Y = A + BX$, where:

$\hat{A} = \bar{Y} - \hat{B}\bar{X}$ where $\bar{Y}$ is the average of the Y's and $\bar{X}$ is the average of the X's,

and

$$\hat{B} = \frac{\sum_{i=1}^{n} x_i y_i - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}$$

In the above example,

$\Sigma X_iY_i = -28.8735$ $\quad \Sigma Y_i = 40.5460$ $\quad \bar{Y} = 8.1092$

$\Sigma X_i = -3.8714$ $\quad \Sigma X_i^2 = 7.5356$ $\quad \bar{X} = -0.7743$

So

$$\hat{B} = \frac{-28.8735 - \frac{(-3.8714)(40.5460)}{5}}{7.5356 - \frac{(-3.8714)^2}{5}}$$

$$\hat{B} = \frac{2.5205}{4.5381}$$

$\hat{B} = 0.5554$

and

$\hat{A} = 8.1092 - (0.5554)(-0.7743)$

$\hat{A} = 8.5392$

Step 4: Calculate the median rank regression estimates of β and η:

$$\hat{\beta} = \frac{1}{\hat{B}} = \frac{1}{0.5554} = 1.80$$

$$\hat{\eta} = e^{\hat{A}} = e^{8.5392} = 5111.25$$

The Weibull equation used to calculate the probability of failure before t cycles is then:

$$F(t) = 1 - e^{-(t/5111.25)^{1.8}}$$

Figure C.1 shows the data plotted on Weibull paper with the least squares line overlaid.

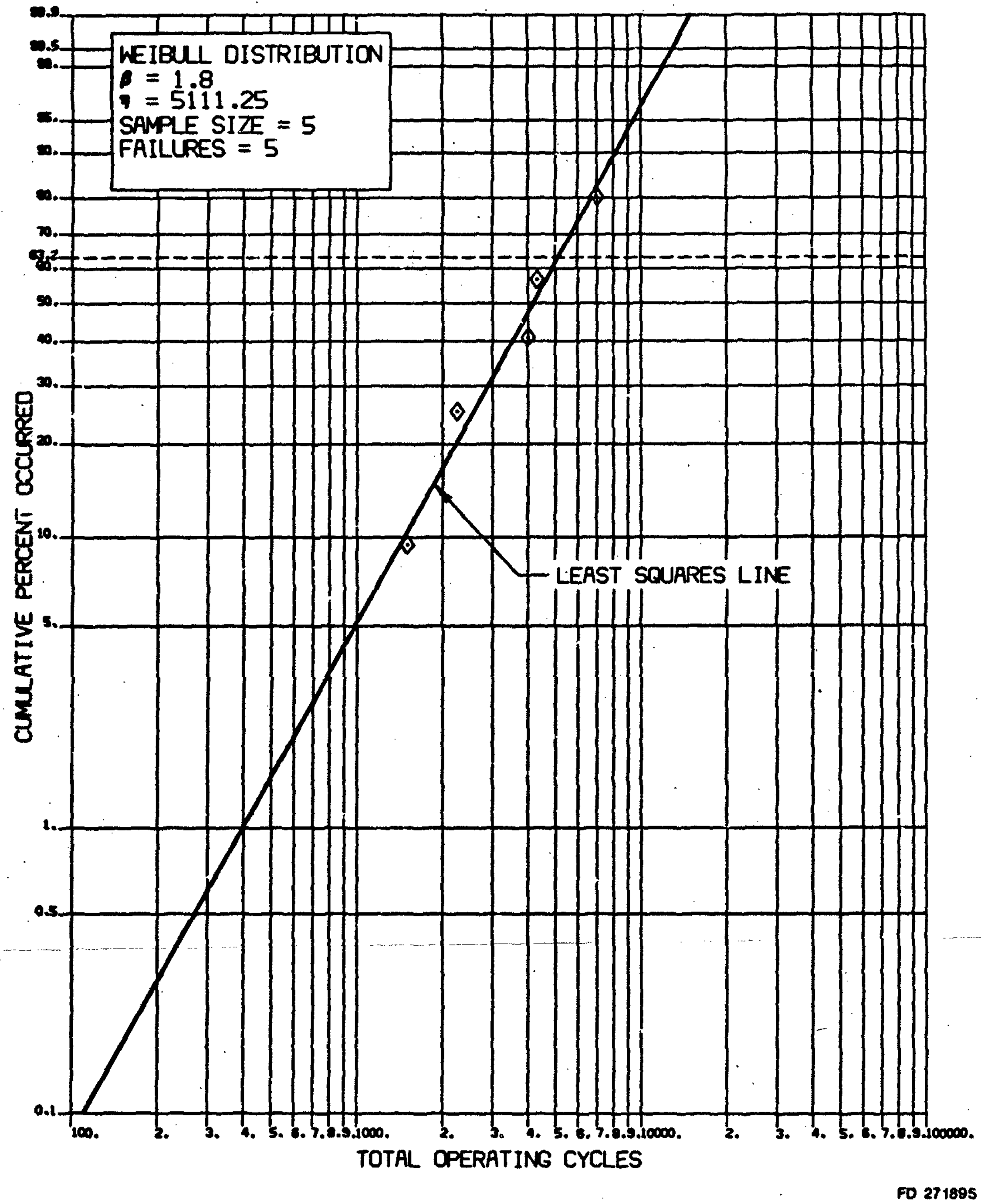


Figure C.1. Example Data With Least Squares Line

APPENDIX D

MAXIMUM LIKELIHOOD METHOD OF WEIBULL ANALYSIS

1. FOREWORD

Weibull analysis consists of "fitting" failure data to the Weibull distribution by estimating the parameters, beta (β) and eta (η). The rank regression (Weibull plot) method was presented in Chapter 2; this Appendix presents an explanation and an example of maximum likelihood Weibull analysis. Details of the maximum likelihood method may be found in Wayne Nelson's text, Applied Life Data Analysis (1982), John Wiley and Sons, New York.

The maximum likelihood Weibull analysis method consists of finding the values of β and η which maximize the "likelihood," of obtaining the observed data. The likelihood of obtaining the observed data, expressed in mathematical form, is a function of the Weibull parameters β and η. Maximum likelihood finds the values of β and η which maximize this mathematical likelihood function.

2. THE LIKELIHOOD FUNCTION

The likelihood function is the mathematical expression of the probability of obtaining the observed data.

When the sample is complete (all units are run to failure), the likelihood function is:

$$L = \prod_{i=1}^{n} f(x_i) = f(x_1)\,f(x_2) \cdots\cdots f(x_n)$$

where n = sample size

$$f(x) = \frac{dF(x)}{dx}$$

and $F(x) = 1 - e^{-(x/\eta)^\beta}$

In reliability terms, F(x) is the probability that a unit will fail before it acquires x units of operating time. F(x) is often called the "unreliability" at time x, and satisfies

$$F(x) = 1 - R(x)$$

where R(x) = reliability at time x.

When the time-to-failure distribution is Weibull,

$$f(x) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^\beta}$$

$$\text{and } L = \prod_{i=1}^{n} \left(\frac{\beta}{\eta}\right)\left(\frac{x_i}{\eta}\right)^{\beta-1} e^{-(x_i/\eta)^\beta}$$

Note that the "likelihood" of the sample failure data $x_1, x_2, \ldots, x_n$ is a function of the Weibull parameters β and η.

The general form of the likelihood function for censored samples (where not every unit has been run to failure) is:

$$L = \prod_{i=1}^{r} f(x_i) \prod_{j=1}^{k} (1 - F(T_j))$$

where

r = number of units run to failure
k = number of unfailed units
$x_1, x_2, \ldots, x_r$ = known failure times
$T_1, T_2, \ldots, T_k$ = operating time on each unfailed unit.

When the time-to-failure distribution is Weibull,

$$L = \prod_{i=1}^{r} \left(\frac{\beta}{\eta}\right) \left(\frac{x_i}{\eta}\right)^{\beta - 1} e^{-(x_i/\eta)^\beta} \prod_{j=1}^{k} e^{-(T_j/\eta)^\beta}$$

3. MAXIMIZING THE LIKELIHOOD FUNCTION

The maximum likelihood method finds the values of β and η which maximize the likelihood function. To find the values of β and η which maximize the Weibull likelihood function, differentiate the logarithm of the likelihood function with respect to β and η, equate the resulting expressions to zero, and simultaneously solve for β and η.

In the complete sample case, the maximum likelihood estimate of β, denoted $\hat{\beta}$, satisfies

$$\frac{\sum_{i=1}^{n} x_i^{\hat{\beta}} \ln x_i}{\sum_{i=1}^{n} x_i^{\hat{\beta}}} - \frac{1}{n} \sum_{i=1}^{n} \ln x_i - \frac{1}{\hat{\beta}} = 0$$

Given the failure times $x_1, x_2, \ldots, x_n$, the maximum likelihood estimate of β is found using iterative procedures.

The maximum likelihood estimate of η is:

$$\hat{\eta} = \left(\frac{\sum_{i=1}^{n} x_i^{\hat{\beta}}}{n} \right)^{1/\hat{\beta}}$$

where $\hat{\beta}$ is the maximum likelihood estimate of β.

When the sample is censored, the maximum likelihood β estimate, $\hat{\beta}$, satisfies

$$\frac{\sum_{i=1}^{N} x_i^{\hat{\beta}} \ln x_i}{\sum_{i=1}^{N} x_i^{\hat{\beta}}} - \frac{1}{r} \sum_{i=1}^{r} \ln x_i - \frac{1}{\hat{\beta}} = 0$$

where N = total sample size

(N = number of failures (r) + number of suspensions (k)).

Units censored at times T_i are assigned the values x_{r+i} T_i. The second term in the equation, $\frac{1}{r}\sum_{i=1}^{r} \ln x_i$, sums the logarithms of the failure times only.

$\hat{\beta}$ is again found using iterative procedures.

Analogous to the complete sample case, the maximum likelihood estimate of η is, in censored samples,

$$\hat{\eta} = \left(\frac{\sum_{i=1}^{N} x_i^{\hat{\beta}}}{r} \right)^{1/\hat{\beta}}$$

4. EXAMPLE

The maximum likelihood method will be illustrated with the censored data listed below.

Cycles	*Status*
1500	Failure
1750	Suspension
2250	Failure
4000	Failure
4300	Failure
5000	Suspension
7000	Failure

The maximum likelihood β estimate, $\hat{\beta}$, is the root of the equation

$$G(\beta) = \frac{\sum_{i=1}^{7} x_i^{\beta} \ln x_i}{\sum_{i=1}^{7} x_i^{\beta}} - \frac{1}{5}\sum_{i=1}^{5} \ln x_i - \frac{1}{\beta} = 0$$

The Weibull plot estimate, 1.8, was used as the initial value of β. This and subsequent estimates of β are listed below with the corresponding value of G(β). (A Fortran subroutine using a modified Newton-Raphson procedure was used to find the value of β giving G(β) = 0.)

	β	G(β)
	1.800	−0.1754
	1.802	−0.1746
	2.179	−0.0255
	2.182	−0.0248
	2.255	−0.0007
	2.256	−0.0005
Maximum Likelihood Estimate of Beta →	2.257	−0.0000

The maximum likelihood estimate of η is

$$\hat{\eta} = \left(\frac{\sum_{i=1}^{7} x_i^{2.257}}{5} \right)^{1/2.257}$$

$$\hat{\eta} = 4900.1.$$

APPENDIX E

WEIBAYES METHODS

1. FOREWORD

Weibayes is a method for constructing a Weibull distribution based on assuming a value of β, the Weibull slope parameter. It is used when there are certain deficiencies in the data (for instance, when operating time has been accumulated, but no failures have occurred). Chapter 4 describes several applications of this method.

The Weibayes equation for η is

$$\eta^* = \left(\frac{\sum_{i=1}^{n} t_i^{\beta}}{r}\right)^{1/\beta} \tag{E.1}$$

where β is the assumed value of the Weibull slope parameter,
n is the number of suspensions or unfailed units in the fleet,
$t_1, t_2, \cdots, t_n$ are the operating times accumulated by units $1, 2, \cdots, n$.
r is the number of failures.

If no failures have occurred, r is assumed to be one, i.e., the first failure is imminent. η^* is then a conservative 63% lower confidence bound on the true value of η. If failures have occurred and Weibayes is used, η^* is the maximum likelihood estimator of the true value of η.

2. DERIVATION OF THE WEIBAYES EQUATION

If no failures have occurred, the Weibayes equation with $r = 1$ gives a conservative 63% lower confidence bound on the true value of η.

The lower bound is derived using two facts from statistics:

1. If $t_1, t_2, \ldots, t_n$ represent failure times drawn from a Weibull population with slope parameter β and characteristic life η, then $t_1^{\beta}, t_2^{\beta}, \ldots, t_n^{\beta}$ represents a random sample from an exponential population with mean life $\theta = \eta^{\beta}$. The exponential cumulative distribution function is $F(t) = 1 - e^{-t/\theta}$.

2. If no failures have occurred in a fleet with n units having operating times $t_1, t_2, \ldots, t_n$, and the units are susceptible to an exponential failure mode, then a conservative $100(1 - \alpha)\%$ one-sided lower confidence limit on the mean life θ is:

$$\theta \geq \frac{\sum_{i=1}^{n} t_i}{-\ln \alpha} \tag{E.2}$$

Conservative means that the true confidence level is unknown, but is at least 100 (1 − α)%.

Thus, if no failures have occurred in a fleet with n units having operating times $t_1, \ldots, t_n$ and the units are susceptible to a Weibull failure mode with known β (and unknown η), then a conservative 100 (1 − α)% one-sided lower confidence limit on $\theta = \eta^\beta$ is:

$$\theta = \eta^\beta \geq \frac{\sum_{i=1}^{n} t_i^\beta}{-\ell n \alpha}$$

$$\text{or } \eta \geq \left(\frac{\sum_{i=1}^{n} t_i^\beta}{-\ell n \alpha}\right)^{1/\beta} \qquad \text{(E.3)}$$

The Weibayes lower bound on η (E.1) is equal to the lower confidence bound in (E.3) with the denominator, $-\ell n\, \alpha = 1.0$. Solving for α, we find:

$$\begin{aligned} -\ell n\ \alpha &= 1.0 \\ \ell n\ \alpha &= -1.0 \\ \alpha &= e^{-1.0} \\ \alpha &= 0.368 \end{aligned}$$

Thus, the Weibayes lower bound on η is a 100 (1 − 0.368)% = 63.2% conservative lower confidence bound for η.

The confidence level can be increased by decreasing α in the denominator of the expression on the right hand side of inequality (E.3). For example, a conservative 90% lower confidence bound on η can be calculated by setting α = 0.10, giving

$$\eta \geq \left(\frac{\sum_{i=1}^{n} t_i^\beta}{-\ell n 0.1}\right)^{1/\beta} \text{ with at least 90\% confidence}$$

$$= \left(\frac{\sum_{i=1}^{n} t_i^\beta}{2.3}\right)^{1/\beta}$$

Note that with β assumed, determining a lower bound for η also determines a lower bound for the Weibull line.

If failures have occurred, and Weibayes is used, η^* is the maximum likelihood estimator of the true value of η. This is shown by finding the value of η that maximizes the Weibull likelihood equations from Appendix D, while assuming that β, the Weibull slope parameter, is known.

These calculations, similar to those discussed in Appendix D, result in the following equation for the maximum likelihood estimator of η (assuming that β is known):

$$\hat{\eta} = \left(\frac{\sum_{i=1}^{n} t_i^{\beta}}{r}\right)^{1/\beta} \qquad \text{(E.4)}$$

Equations E.1 (Weibayes) and E.4 (maximum likelihood) are identical, demonstrating that the Weibayes equation yields the maximum likelihood estimator of η, when failures have occurred and β is known.

APPENDIX F

MONTE CARLO SIMULATION STUDY — ACCURACY OF WEIBULL ANALYSIS METHODS

1. FOREWORD

Safety considerations in aerospace operations require corrective action based on very small samples of failure data. As this is unusual compared to other Weibull applications, Monte Carlo simulation was used to study the accuracy of Weibull analysis when applied to data from a fleet of several thousand successfully operating units and very few service failures (three to ten failures). Of prime importance were the accuracy and precision of the risk forecasts, the β estimates, and the B.1 life estimates. Two methods of Weibull analysis were considered and compared: median rank regression and maximum likelihood. These two methods are among the most commonly used for estimating β and η with multiply censored data (i.e., data containing both failures and suspensions, with the suspensions and failure times intermixed). Median rank regression is covered in Chapter 2 and Appendix C. Maximum likelihood methods are introduced in Appendix D. Details of the Monte Carlo simulation method are discussed later in this appendix.

Figure F.1 illustrates the meaning of the accuracy and precision of Weibull parameter estimates. A simulated fleet with 2000 operating units was introduced to a Weibull failure mode with $\beta = 3$ and $\eta = 13{,}000$ hours. When the fleet experienced its 5th failure, a Weibull analysis was performed. The estimated values of β and η were stored, the fleet was re-created and re-introduced to the Weibull failure mode. A Weibull analysis was again done at the time of the 5th failure. The β and η estimates were stored, and the process was repeated many times, for a total of 1000 estimates of β and η under the simulated circumstances. Figure F.1 is a histogram of the β estimates. It shows that 294 of the 1000 estimates of β were between 0.5 and 2.5. (No estimates were below 0.5). In 392 out of 1000 simulation trials, β was estimated between 2.5 and 4.5. "Accuracy" refers to the difference between the "typical" β estimate and the true value of β. In this case, the median β estimate is 3.5 (50% of the estimates are below 3.5, 50% are above 3.5) and the true value of β is 3.0: the β estimates are quite accurate with 5 failures. Precision refers to the variability in the β estimates, and is normally measured by the standard deviation. The more the β estimates deviate from their mean value, the higher is their standard deviation.

The standard deviation of the 1000 β estimates $\beta_1, \beta_2, \ldots, \beta_{1000}$, is calculated as:

$$S = \sqrt{\frac{\sum_{i=1}^{1000} (\beta_i - \beta_{avg})^2}{999}}$$

where

$$\beta_{avg} = \frac{\beta_1 + \beta_2 + \ldots + \beta_{1000}}{1000}$$

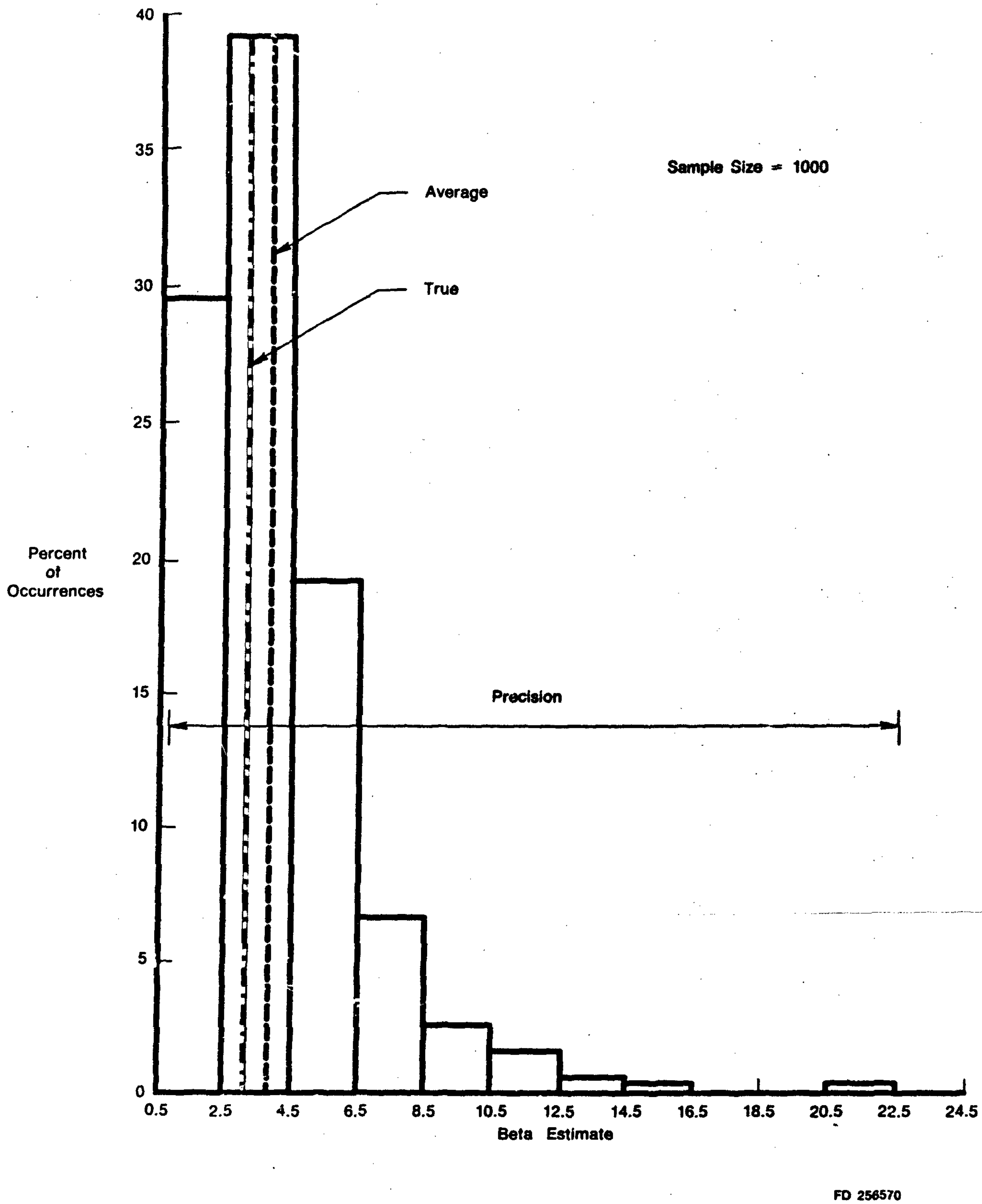


Figure F.1. Beta Estimates: True Beta = 3, Five Failures

2. BETA (β) ESTIMATES

The accuracy of the maximum likelihood (ML) and median rank regression (MRR) β estimates, calcuated with a small number of failures and a large number of unfailed units, is illustrated in Figure F.2.

The standard deviations of the small sample β estimates are shown in Figure F.3.

The following can be seen from Figures F.2 and F.3:

1. Both methods of analysis, median rank regression and maximum likelihood, tend to overestimate β. When the failure data are plotted on Weibull paper, the slope is generally too steep.

2. The accuracy and precision of the β estimates improve as the number of failures increases.

3. Both methods of analysis produce estimates of comparable accuracy. Maximum likelihood estimates are noticeably more accurate for $\beta = 5.0$, especially with very few failures.

4. Maximum likelihood β estimates are more precise than median rank regression estimates, especially when the data contain as few as three to five failures. In the three-failure case, maximum likelihood estimates have between 44% and 68% less variability than median rank regression estimates.

3. B.1 LIFE ESTIMATES

The B.1 life is frequently used as a design criteria. Its estimate from field data is often compared to the predicted or design B.1 life, so its accurate estimation is important.

Figure F.4 illustrates the accuracy of the B.1 life estimates over the range of the study.

Figure F.5 shows the B.1 life standard deviations as a function of sample size (3 to 10 failures, 1000 to 2000 unfailed units).

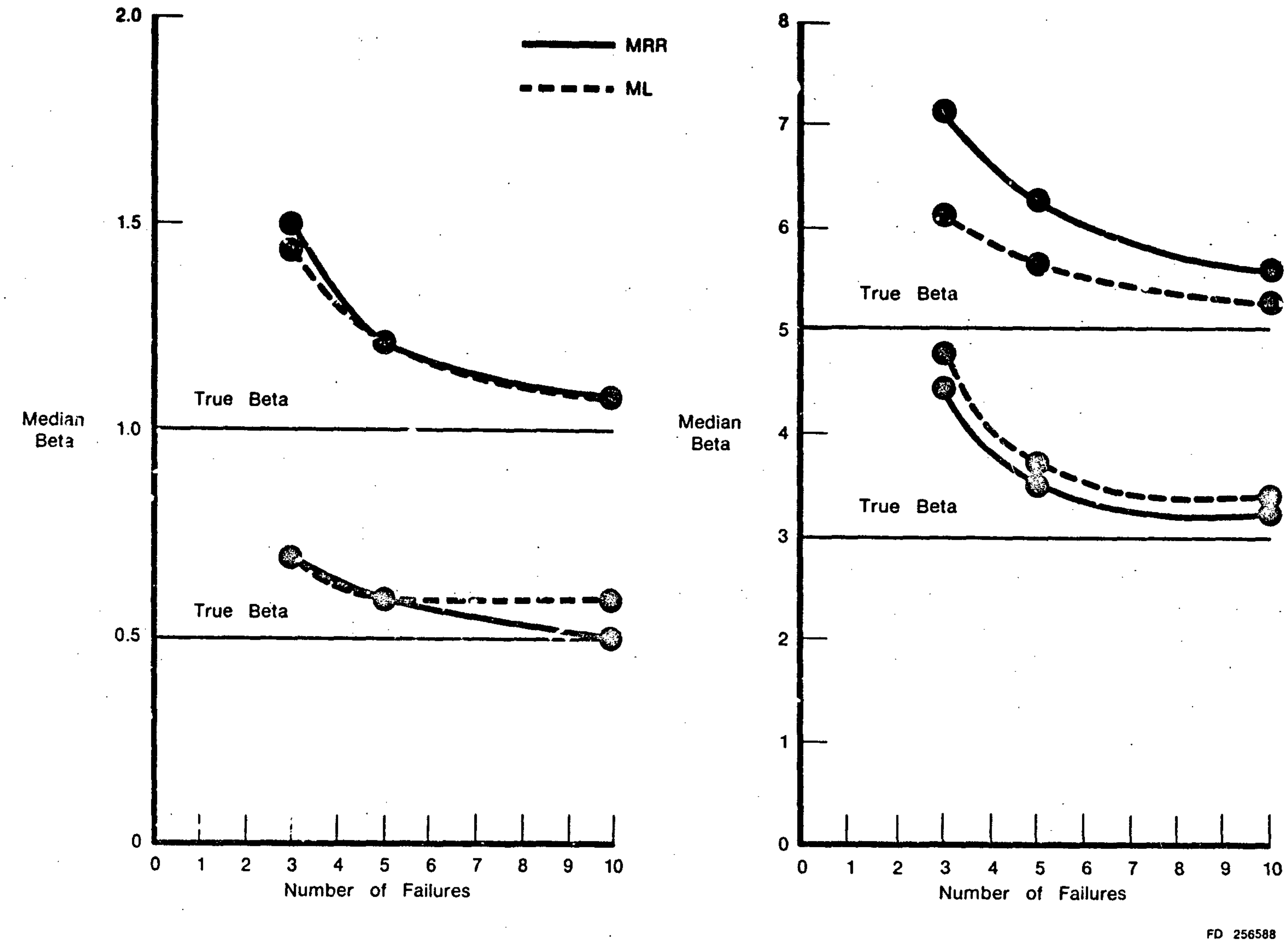


Figure F.2. Beta Accuracy

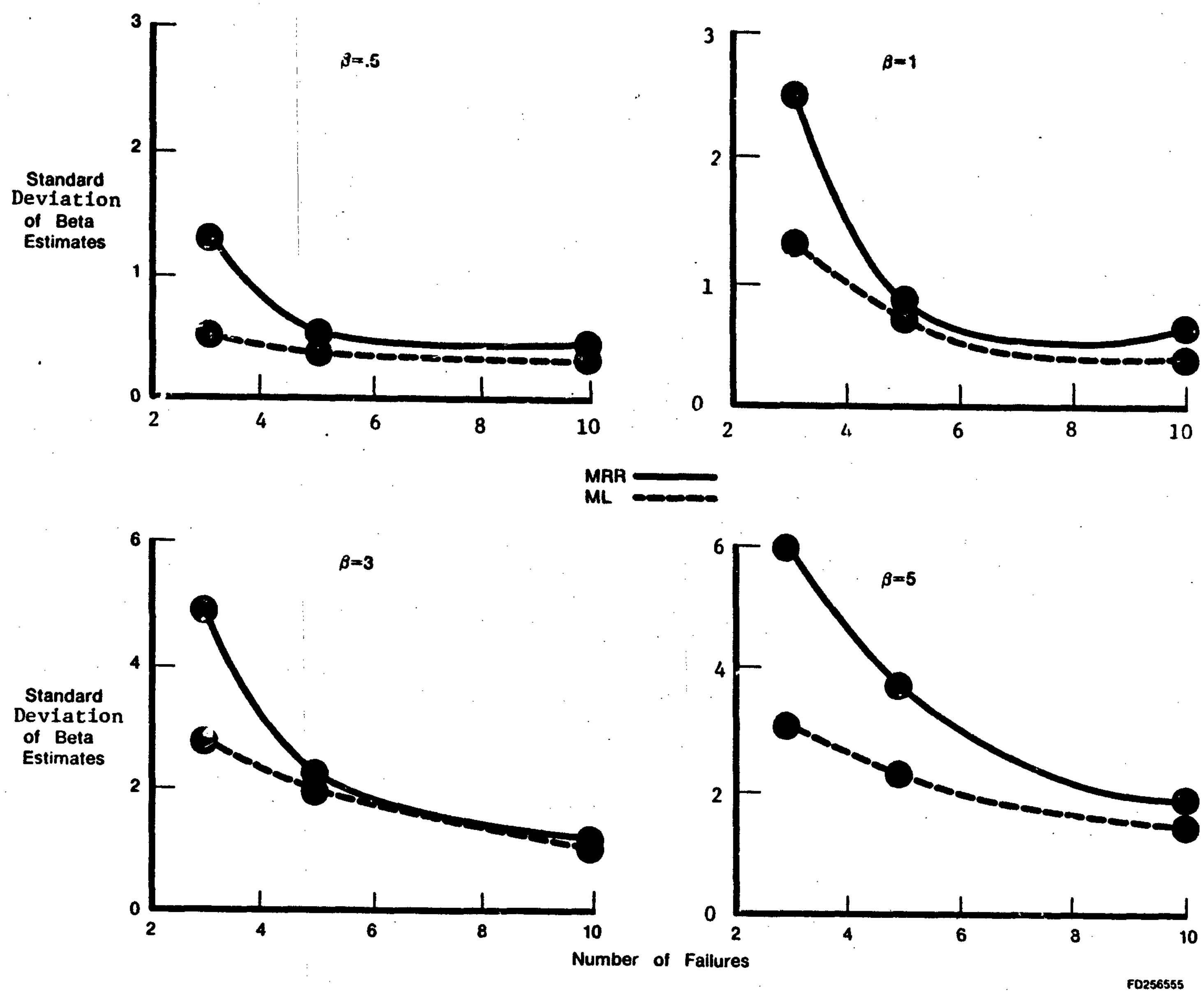


Figure F.3. Beta Precision

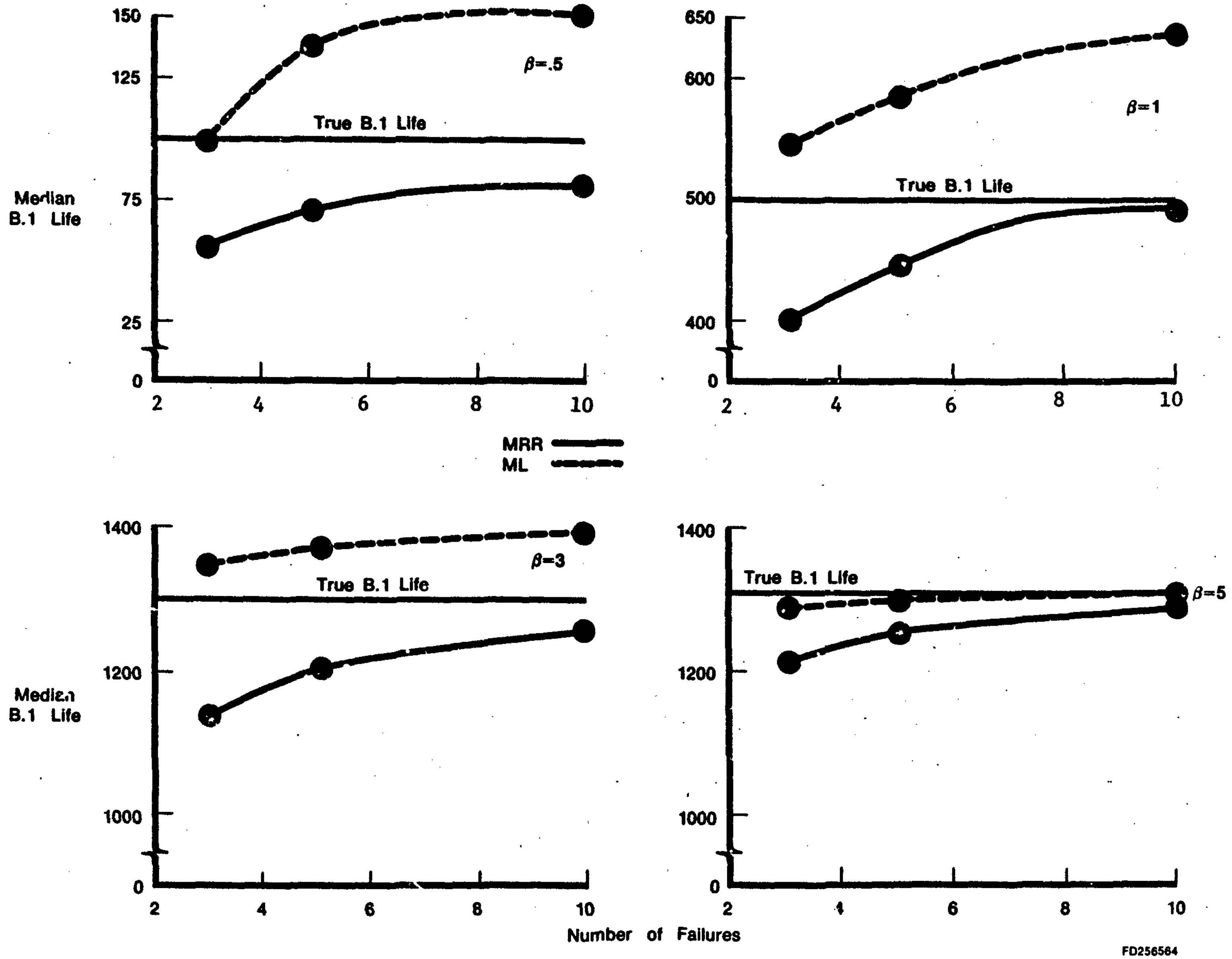


Figure F.4. B.1 Life Accuracy

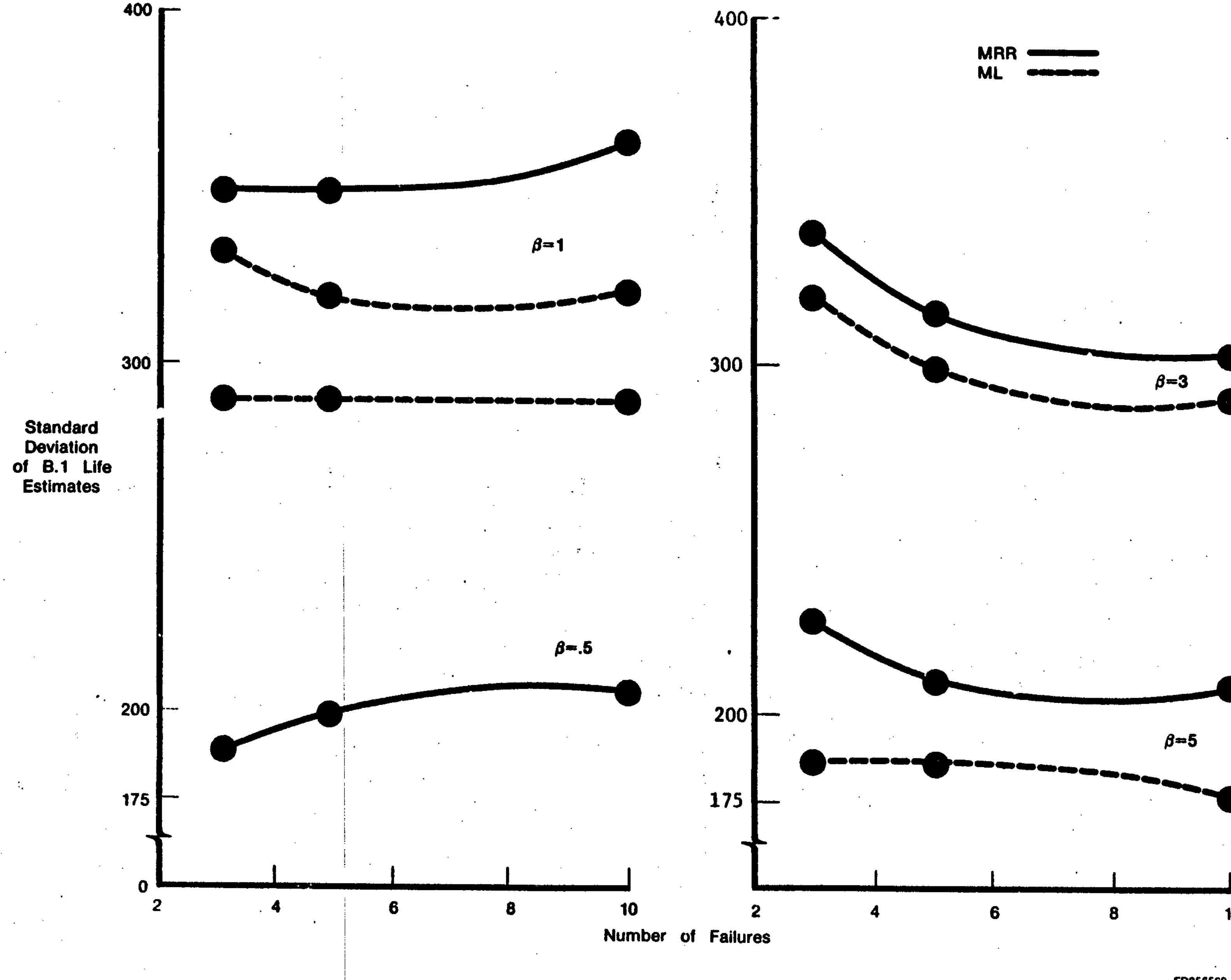


Figure F.5. B.1 Life Precision

Figures F.4 and F.5 indicate that:

1. Median rank regression estimates of the B.1 life are typically conservative: the estimated B.1 life is more often than not less than the true B.1 life.

2. The accuracy of the median rank regression B.1 life estimates improves as the number of failures increases.

3. For β = 0.5, 1, and 3, the maximum likelihood method typically overestimated the B.1 life (ML overestimated the B.1 life in approximately 58% of the simulation trials, for β = 0.5, 1, and 3.0 MRR overestimated the B.1 life in ~ 40% of the simulation trials).

4. For β = 5, maximum likelihood typically underestimated the B.1 life (ML underestimated the B.1 life in 52% of the simulation trials, vs. 58% for MRR).

5. The precision of the B.1 life estimate does not necessarily improve as the number of failures increases from three to ten.

4. ETA (η) ESTIMATES

The medians of the characteristic life estimates from simulated fleets with few failures are shown in Figures F.6 and F.7.

Table F.1 contains the standard deviations of the characteristic life estimates.

TABLE F.1. STANDARD DEVIATIONS OF THE CHARACTERISTIC LIFE ESTIMATES

			Standard Deviations of the Eta Estimates		
Type Estimates	*True Beta*	*True Eta*	*3 Failures*	*5 Failures*	*10 Failures*
MRR	0.5	100,000,000	4.8×10^{22}	4.3×10^{20}	6.5×10^{16}
ML			2.7×10^{14}	1.9×10^{14}	3.0×10^{10}
MRR	1.0	500,000	8.8×10^{13}	1.3×10^{12}	1.2×10^{9}
ML			4.6×10^{13}	9.2×10^{8}	8.1×10^{6}
MRR	3.0	13,000	320,000	76,000	21,000
ML			27,000	15,000	6,300
MRR	5.0	5,175	8,200	5,900	2,200
ML			3,300	3,000	1,900

MRR = Median Rank Regression
ML = Maximum Likelihood

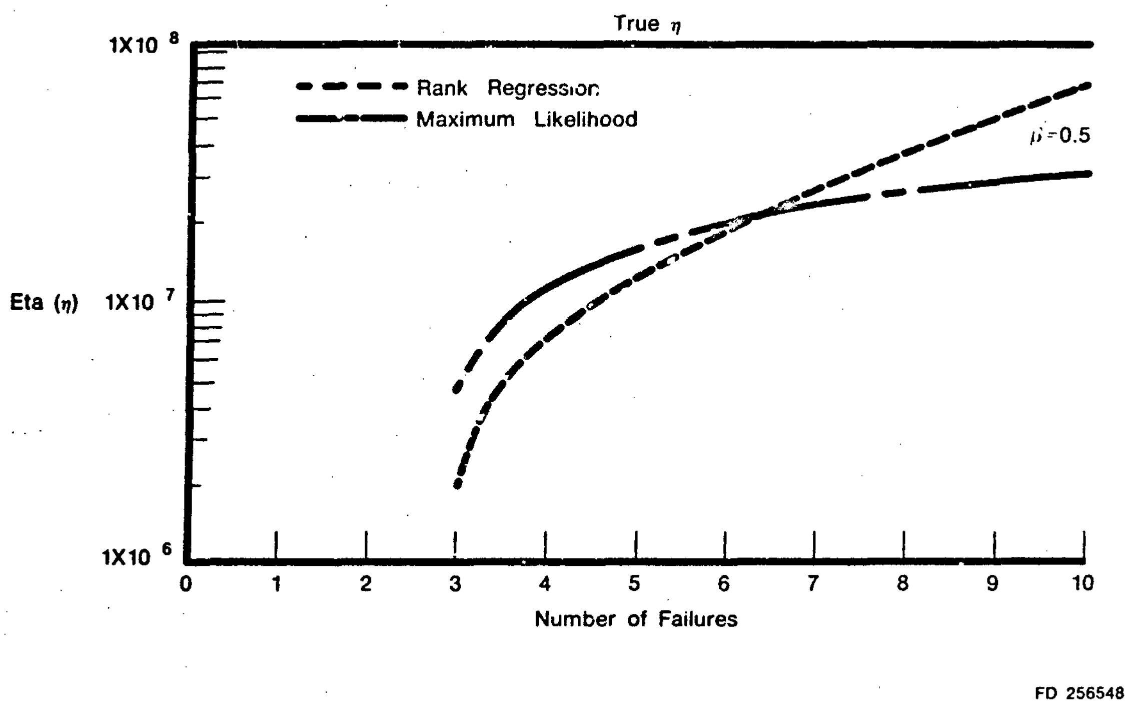


Figure F.6. Eta Accuracy Median Characteristics Life Estimates — A

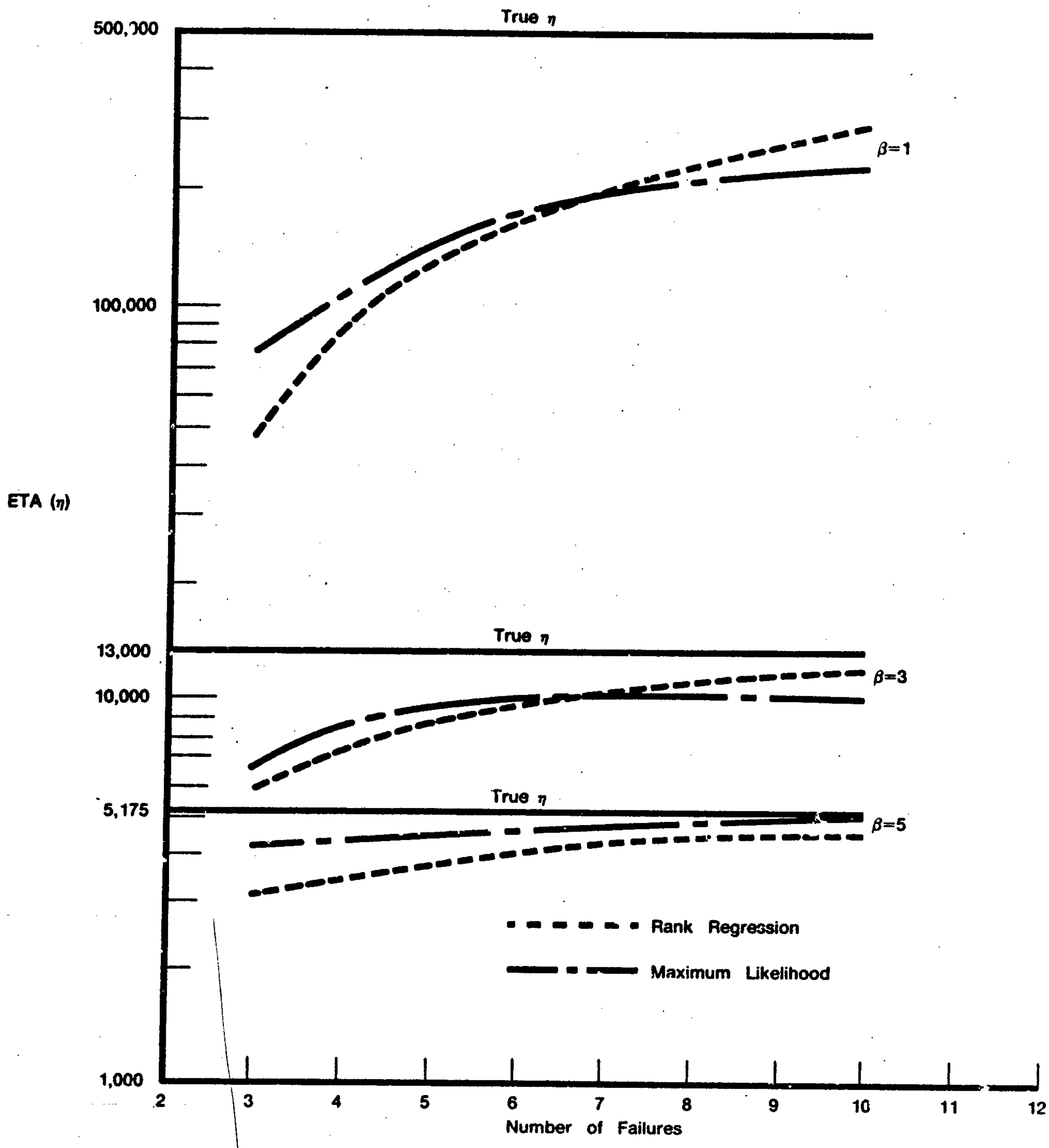


Figure F.7. Eta Accuracy — Median Characteristics Life Estimates — B

Table F.1 and Figures F.6 and F.7 indicate that:

1. The median rank regression (MRR) and maximum likelihood (ML) estimates are conservative: η is typically underestimated.

2. The accuracy of both types of estimates improves as the number of failures increases from 3 to 10.

3. The precision of the characteristic life estimates improves as the number of failures increases.

4. The standard deviations of the η estimates are extremely large, reflecting the presence of several extremely large estimates of η at each simulated condition.

5. RISK FORECASTS

This section addresses the accuracy and precision of risk forecasts made when there are only 3 to 10 failures and 1000 to 2000 suspensions in a fleet (a risk forecast is a prediction of the number of failures expected to occur over a period of calendar time). Methods for constructing risk forecasts from multiply-censored life data can be found in Chapter 3.

The accuracy and precision of the risk forecasts are assessed by advancing a simulated fleet through a known Weibull failure mode to the time of its 3rd, 5th, and 10th failures. A Weibull analysis is done and the risk forecast is made, up to 12 months into the future. The fleet is then advanced 12 months further through the Weibull failure mode using an average military aircraft utilization rate. The risk forecast is then compared to the actual number of failures caused by the fleet's additional 12 month advance. This procedure is repeated 100 times with 100 different simulated fleets to assess the variability and accuracy of the risk estimates.

5.1 RISK FORECAST ACCURACY

Table F.2 contains the medians of the maximum likelihood and rank regression forecasts for 0, 6, and 12 months into the future. The entry under "0" months ahead indicates the cumulative number of failures expected 'to date' (at the time the analysis is performed). Thus, when $\beta = 3$ and there are three failures in the fleet, there are $5 - 3 = 2$ additional failures expected over the 12 month interval following the occurrence of the third failure. The rank regression method predicts an "average" of $11 - 5 = 6$ additional failures, and maximum likelihood predicts an "average" of $6 - 3 = 3$ additional failures.

TABLE F.2. RISK FORECAST ACCURACY

$\beta = 0.5$

No. Failures	Months Ahead	Median Risk	Median MRR Forecast	Median ML Forecast
3	0	3	4	3
	6	3	6	4
	12	4	7	4
5	0	5	6	5
	6	5	7	5
	12	6	7	6
10	0	10	11	10
	6	10	11	10
	12	10	11	11

$\beta = 1$

3	0	3	4	3
	6	3	5	4
	12	4	6	4
5	0	5	7	5
	6	5	7	6
	12	6	8	6
10	0	10	11	10
	6	11	11	11
	12	11	12	11

$\beta = 3$

No. Failures	Months Ahead	Median Risk	Median MRR Forecast	Median ML Forecast
3	0	3	5	3
	6	4	8	4
	12	5	11	6
5	0	5	7	5
	6	6	9	6
	12	7	11	8
10	0	10	12	10
	6	12	14	12
	12	14	17	15

$\beta = 5$

No. Failures	Months Ahead	Median Risk	Median MRR Forecast	Median ML Forecast
3	0	3	6	3
	6	4	12	5
	12	7	22	8
5	0	5	8	5
	6	7	14	8
	12	10	25	12
10	0	10	13	10
	6	14	19	14
	12	19	28	21

From Table F.2, it is seen that:

1. Rank regression risk estimates tend to be very conservative - they overpredict the number of failures. Maximum likelihood risk forecasts are only slightly conservative.

2. Forecasts based on maximum likelihood Weibull analyses are always more accurate than forecasts based on rank regression Weibull analyses.

3. The accuracy differences between the two methods increases as β increases. Maximum likelihood methods are much more accurate than rank regression with $\beta = 5$; they are only marginally more accurate when $\beta = 0.5$.

5.2 RISK FORECAST PRECISION

Table F.3 presents the standard deviations of the risk forecasts 0, 6, and 12 months into the "future". (The risk forecast 0 months into the future is the expected number of failures to date.)

TABLE F.3. RISK FORECAST STANDARD DEVIATIONS

$\beta = 0.5$

No. Failures	*Months Ahead*	*MRR Forecast*	*ML Forecast*
3	0	22.0	0.003
	6	52.0	0.98
	12	91.0	2.3
5	0	3.8	0.002
	6	4.8	0.45
	12	6.1	0.91
10	0	2.8	0.004
	6	3.1	0.22
	12	3.3	0.43

$\beta = 1$

No. Failures	*Months Ahead*	*MRR Forecast*	*ML Forecast*
3	0	73.0	0.002
	6	94.0	0.84
	12	112.0	2.3
5	0	34.0	0.002
	6	50.0	0.59
	12	67.0	1.4
10	0	3.2	0.006
	6	3.6	0.24
	12	4.1	0.49

TABLE F.3. RISK FORECAST STANDARD DEVIATIONS

$\beta = 3$

No. Failures	Months Ahead	MRR Forecast	ML Forecast
3	0	14.0	0.006
	6	19.0	1.2
	12	24.0	4.4
5	0	7.4	0.006
	6	10.0	1.1
	12	14.0	3.3
10	0	4.0	0.011
	6	5.1	0.79
	12	6.5	1.9

$\beta = 5$

No. Failures	Months Ahead	MRR Forecast	ML Forecast
3	0	11.0	0.007
	6	37.0	1.4
	12	88.0	5.3
5	0	4.7	0.011
	6	14.0	1.7
	12	36.0	5.8
10	0	4.4	0.015
	6	12.0	1.5
	12	29.0	4.4

Table F.3 shows that:

1. The rank regression forecasts vary substantially (up to 50 times more than forecasts based on maximum likelihood Weibull analyses).

2. Maximum likelihood forecasts are far more precise than rank regression forecasts, over the entire scope of this study.

3. As the number of failures used in the Weibull analysis increases, the precision of the resulting risk forecasts improves.

6. MONTE CARLO SIMULATION

The Monte Carlo simulation consists of input, processing, and output segments. The components of each segment are listed in Table F.4.

TABLE F.4. COMPONENTS OF MONTE CARLO SIMULATOR

Input	*Processing*	*Output*
1. Fleet size	1. Fleet construction	1. Weibull parameter estimates
2. Fleet age distribution	2. Fleet aging	2. One-year-ahead risk forecasts
3. Production schedule	3. New member addition	3. Actual one-year ahead failure counts
4. Usage rate	4. Random failure time generation	
5. True Weibull parameters	5. Parameter estimation	
6. Random numbers	6. Risk forecasting	

The Monte Carlo simulation procedure is illustrated in Figure F.8.

Fleet characteristics and Weibull failure mode parameters are input into the simulator. The fleet characteristics are (1) the number of units in the fleet; (2) the fleet age distribution; (3) the production schedule; and (4) the per-unit monthly usage rate. The age distribution is assumed to be normal, with 1000 to 2000 units in the fleet. New members are brought into the fleet according to a production schedule and utilized at a rate typically experienced with Pratt & Whitney Aircraft military gas turbine engines. The simulator generates failures among the units in the fleet according to the Weibull failure mode input. Slope parameters (βs) of one-half, one, three, and five were chosen to represent infant mortality, random, wearout, and rapid wearout failure modes, respectively. See Figure F.9.

The processing segment includes constructing and aging the fleet, incorporating new members, generating random failure times, estimating the Weibull parameters, and forecasting the additional number of failures expected in the year ahead. Failure times are generated for each member of the fleet using random numbers and the input Weibull parameters. The fleet is aged until 3, 5, and 10 failures occur. Weibull analyses are performed at these times, and the year-ahead forecasts are made. This process continues until 10 failures occur in the fleet. The fleet is then reconstructed, and the process begins again. For the rank regression estimators this process was repeated 1000 times for each failure mode considered. Cost considerations limited this number to 100 for the maximum likelihood estimators.

Parameter estimates, year-ahead failure forecasts, and actual year-ahead failure counts are output. The error in the estimation method is reflected in the differences between the parameter estimates and the actual Weibull parameters input to the simulator. The forecasting error is simply the difference between the actual and forecast failure counts.

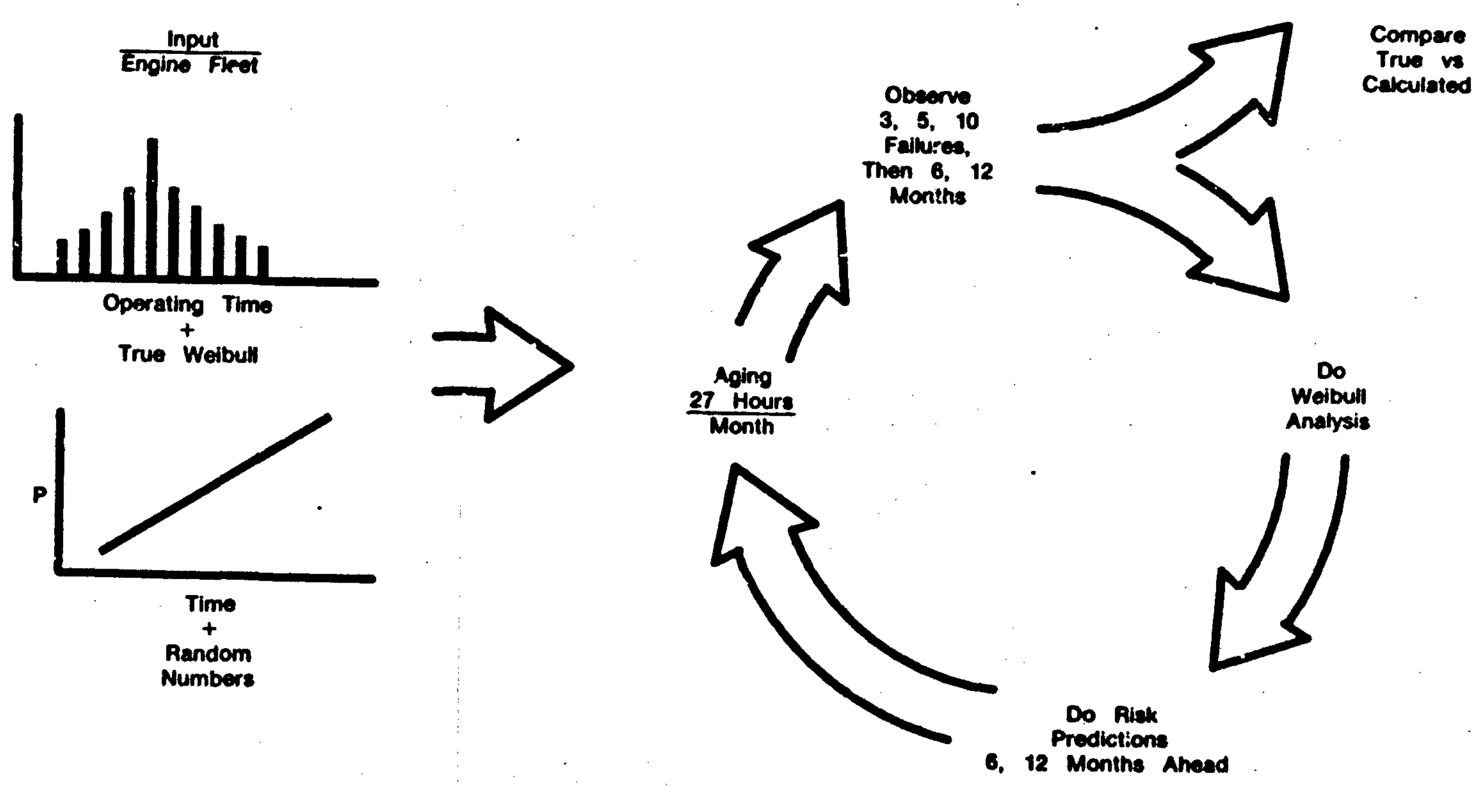


Figure F.8. Monte Carlo Simulation Procedure

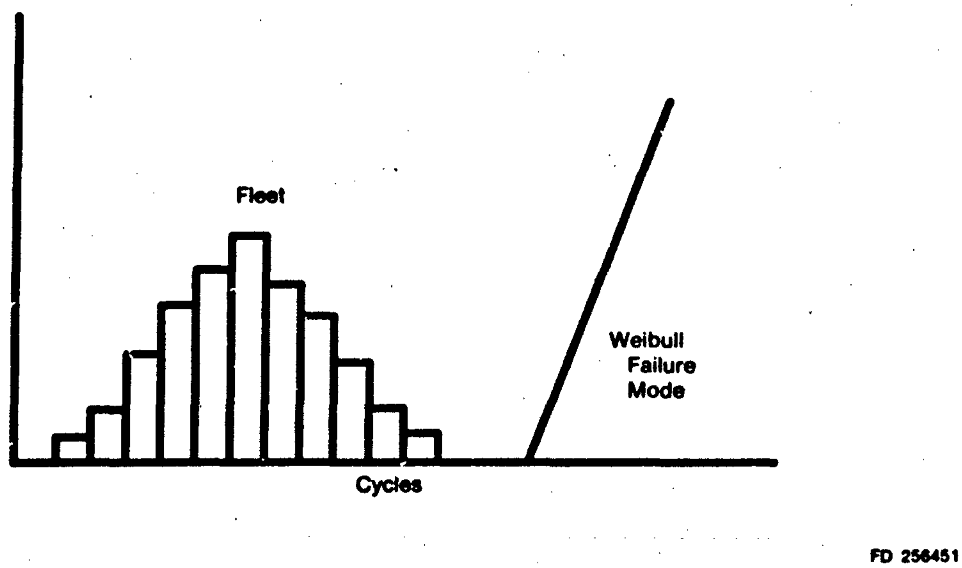


Figure F.9. Initial Conditions of Simulator

APPENDIX G

RANK REGRESSION METHOD VS MAXIMUM LIKELIHOOD METHOD OF WEIBULL ANALYSIS

This appendix contains a summary of the strengths and weaknesses (and general comments) of the rank regression method and the maximum likelihood method of Weibull analysis.

1. Rank regression provides a graphical display of the data. This helps to identify instances of:

   a. The Weibull distribution not fitting well to the data (suggesting perhaps another distribution, like the log-normal),

   b. More than one failure mode affecting the units,

   c. Data needing a t_o correction,

   d. Outliers in the data,

   e. Batch problems.

   Maximum likelihood does not provide a graphical display of the data.

2. Rank regression provides accurate estimates of "low" percentiles like the B.1 life under the conditions simulated in Appendix F, even for small sample sizes. "Low" percentiles refer to percentiles close to the time of the first failure. Maximum likelihood "lower" percentile estimates have a slight positive bias with small numbers of failures and a large number of suspended items.

3. Rank regression risk forecasts are conservative (overestimate the risk) and less precise, when computed with few failures and a couple of thousand suspensions. Maximum likelihood risk forecasts are more accurate and precise than are rank regression risk forecasts with small failure samples.

4. Both rank regression and maximum likelihood tend to overestimate β with small failure samples. (The slope on the Weibull plot is too steep.) This positive bias decreases as the number of failures increases.

5. Confidence intervals on the Weibull parameters β and η based on rank regression estimates are not available. Statistical hypotheses about β and η (e.g. Is $\beta = 1$, implying the failure mode is random or memoryless?) cannot be tested using these estimates. Exact or approximate (large sample) confidence intervals on the Weibull parameters β and η, based on the maximum likelihood estimates, are available for all commonly occurring forms of censored data. Statistical hypotheses about β and η can be tested using these estimates.

APPENDIX H

WEIBULL PARAMETER ESTIMATION COMPUTER PROGRAMS

Enclosed are program listings for estimating the Weibull distribution for data containing both complete and censored samples. Two programs are provided, one in BASIC and one in FORTRAN, both of which were developed for and will run on a TRS80 Model I or III microcomputer. They take approximately 10K bytes of RAM; but, in the case of the FORTRAN program, will require a FORTRAN compiler, at least one disk drive, and 48K memory.

Both programs will estimate the parameters of a Weibull distribution by the rank regression and maximum likelihood technique. Hence, they can be used to solve many of the problems and examples in this Handbook. Of course, use of the programs beyond this Handbook can be made; however, first the programs should be checked thoroughly by the user.

The programs run in the "immediate" mode, i.e., the user is prompted for input, with no printout options. The programs can be used to generate estimates of the Weibull distribution parameters for samples of size less than or equal to 100. If more than 100 failures are to be input, the dimension statements at the front of both programs should be increased.

If a histogram of a suspension population is available, along with the failure times of interest, a Weibull analysis can be produced for quite large samples. The histogram is input bar by bar and the programs assign the midpoint time of each bar to all of the units in the bar.

Of special note is the maximum likelihood parameter estimation capability in both programs. These programs illustrate the solution of the maximum likelihood equation (Appendix D.3) by the use of the Newton-Raphson iteration procedure.

It should be noted that, for those example cases that are run with histograms from this Handbook, the same answers as given in the Handbook may not be achieved. The parameter estimates should be close, however, and will only be different because of the histogram input in the case of these programs, where the individual times on each suspended unit were used for the examples in the Handbook.

Listing 1. BASIC Program for Weibull Parameter Estimation

```
10  DIM IS(100),RA(100),A(100),RO(100),TT(100),XZ(100)
20  DIM VX(100),IN(100),TI(100),V(100),IZ(100),X(100)
30  IP=0
40  IM=0
50  PRINT"ARE YOU INPUTTING A HISTOGRAM OF SUSPENSIONS?"
60  PRINT"ANSWER Y OR N"
70  INPUT A$
80  IF A$="N" GOTO 310
90  IP=1
100 PRINT"AN INTERVAL SIZE OF 50 IS ASSUMED"
110 PRINT"O.K.?....ANSWER Y OR N"
120 INPUT A$
130 IF A$ < > "N" GOTO 170
140 PRINT"PLACE THE INTERVAL SIZE YOU WILL USE IN"
150 PRINT"CC 1-10 W/DECIMAL"
160 INPUT PE
170 PRINT"PLACE THE NUMBER OF ELEMENTS IN EACH INTERVAL OF"
180 PRINT"THE HISTOGRAM IN CC 1-10 W/DECIMAL"
190 PRINT"USE -99. TO INDICATE THE END"
200 M=0
210 M=M+1
220 INPUT XI
230 IF XI = -99. GOTO 250
240 IN(M)=XI: GOTO 210
250 MM=M-1
260 FOR I=1 TO MM: PRINT IN(I); : NEXT I:PRINT
270 IM=0
280 FOR J=1 TO MM
290 TI(J)=PE/2.+(J-1)*PE
300 IM=IM+IN(J): NEXT J
310 PRINT"INPUT THE FAILURE DATA AND SUSPENSIONS IN"
320 PRINT"CC 1-10 W/DECIMAL... USE -99999. TO INDICATE"
330 PRINT"THE END OF DATA (NEGATIVES INDICATE SUSPENSIONS,"
340 PRINT"UNLESS A HISTOGRAM WAS INPUT)"
350 I=0
360 I=I+1
370 INPUT A(I)
380 IF A(I) =0.0 GOTO 410
390 IF A(I) = -99999. GOTO 430
400 GOTO 360
410 I=I-1
420 GOTO 360
430 N=I-1
440 BN=N+IM
450 FOR J=1 TO N: AW=A(J): V(J)=ABS(AW): NEXT J
460 GOSUB 5000
470 FOR I=1 TO N: IU=IZ(I): X(I)=A(IU): NEXT I
480 FOR I=1 TO N: A(I)=X(I): NEXT I
490 B1=BN+1: DJ=1.0: BJ=0.0: M=0: SX=0.0: SY=0.0: XX=0.0: YY=0.0: XY=0.0
500 PRINT"PT.      DATA      ORDER      MEDIAN RANK"
510 FOR K=1 TO N
```

Listing 1. BASIC Program for Weibull Parameter Estimation (Continued)

```
520  IM  0
530  IF IP  0 GOTO 580
540  FOR J=1 TO MM
550  IF TI(J) < A(K) THEN IM=IM+IN(J)
560  NEXT J
570  IS(K)=!M
580  BK=IM+K
581  IF IP=1 THEN BK=BK-1.0
590  IF IP=1 AND K=1 THEN DJ=(B1-BJ)/(B1-BK)
600  !F K=1 GOTO 630
601  IF IP=0 GOTO 630
610  IF IS(K)=IS(K-1) GOTO 630
620  DJ=(B1-BJ)/(B1-BK)
630  IF A(K) < 0.0 GOTO 660
640  IF A(K) =0.0 GOTO 840
650  IF A(K) > 0.0 GOTO 670
660  DJ=(B1-BJ)/(B1-BK): GOTO 720
670  BJ=BJ+DJ: RO(K)=BJ: RA(K)=(RO(K)-.3)/(BN+.4)
680  X1=LOG(A(K)):YP=1./(1.-RA(K)):Y=LOG(LOG(YP)):YX(K)=Y
681  REM PRINT: "BJ=";BJ;"DJ=";DJ;"BK=";BK;"B1=";B1
690  PRINT K,A(K),RO(K),RA(K)
700  M=M+1
710  SX=SX+X1:XX=XX+X1*X1:SY=SY+Y:YY=YY+Y*Y:XY=XY+X1*Y
720  NEXT K
730  GM=M
740  BE=(GM*YY-SY*SY)/(GM*XY-SX*SY)
750  AL=(BE*SX-SY)/GM:AV=AL/BE:AV=EXP(AV):ST=BE
760  PRINT "BETA=";ST;" ETA=";AV
770  R=0.0
780  IF (XX-SX*SX/GM) < 0.0 COTO 820
790  XN=XY-SX*SY/GM
800  DE=SQR((XX-SX*SX/GM)*(YY-SY*SY/GM))
810  R=XN/DE
820  RQ=R*R
821  IF RQ >1.0 THEN RQ=1.0
830  PRINT "R= ";R;" R SQUARE=";RQ
831  PRINT"DO YOU WISH TO DO MAXIMUM LIKELIHOOD ESTIMATION?"
832  PRINT"ANSWER Y(ES) OR N(O)"
833  INPUT A$
834  IF A$="N" GOTO 840
835  NF=0: PRINT"PLEASE BE PATIENT..IT'S ITERATING"
836  GOSUB 2000
840  END
2000 FOR I=1 TO N
2010 TT(I)=ABS(A(I))
2020 IF A(I)<0.0 GOTO 2050
2030 NF=NF+1
2040 XZ(NF)=A(I)
2050 NEXT I
2060 OT=.0001: NL=100: XB=BE: YA=.001: NC=0: DX=.001: DY=.01
2130 GOSUB 3000: YB=AU
```

Listing 1. BASIC Program for Weibull Parameter Estimation (Continued)

```
2135  GOSUB 4000: XB=BB#
2136  ON JK GOTO 2130,2150,2140,2150
2137  GOTO 2130
2140  PRINT"ITERATION FAILURE"
2145  PRINT"BETA=";XB;" LN MAXIMUM LIKELIHOOD=";YB
2147  RETURN
2150  BL=XB: SU=0.0: RN=NF: FOR I=1 TO N: SU=SU+TT(I)[BL: NEXT I
2160  IF IP=0 GOTO 2250
2170  FOR I=1 TO MM:SU=SU+IN(I)*TI(I)[BL: NEXT I
2250  SU=SU/RN:TL=SU[(1.0/BL)
2260  PRINT"MAXIMUM LIKELIHOOD ESTIMATES FOLLOW"
2270  PRINT"BETA=";BL;" ETA=";TL
2280  RETURN
3000  S1#=0.0 : S2#=0.0 : S3#=0.0
3010  IF XB>15. OR XB < =0.0 THEN XB=0.1
3020  FOR I=1 TO N
3021  PO#=TT(I)[XB
3022  S1#=S1#+PO#
3023  S2#=S2#+LOG(TT(I))*PO#
3030  NEXT I
3040  IF IP=0 GOTO 3130
3050  FOR I=1 TO MM : PO#=TI(I)[XB: S1#=S1#+IN(I)*PO#
3060  S2#=S2#+IN(I)*LOG(TI(I))*PO#: NEXT I
3130  FOR I-1=TO NF: S3#=S3#+LOG(XZ(I)): NEXT I
3140  AU=(S2#/S1#)-(S3#/NF)-(1.0/XB)
3150  RETURN
4000  JK=1: BB#=XB
4010  IF (ABS((YA-YB)/YA)-OT) < = 0.0 GOTO 4290
4020  IF (NC-1) < = 0 GOTO 4040
4030  GOTO 4090
4040  DX#=BB#
4050  DY#=YA-YB
4060  NC=NC+1
4070  BB#=BB#*1.02
4080  RETURN
4090  IF NC > NL GOTO 4300
4100  X2#=BB#
4110  D2#=YA-YB
4120  IF ABS(D2#-DY#) < .00001 GOTO 4320
4130  BB#=X2#-D2#*(X2#-DX#)/(D2#-DY#)
4150  IF BB# < = 0.0 GOTO 4250
4160  IF BB#<X2# GOTO 4190
4170  IF BB#=X2# GOTO 4240
4180  IF BB#>X2# GOTO 4210
4190  IF BB#/X2# > =.6 GOTO 4250
4200  BB#=X2#*.75: GOTO 4250
4210  IF BB#/X2# < 1.4 GOTO 4250
4220  BB#=X2#*1.25
4230  GOTO 4250
4240  BB#=X2#*1.02
4250  DX#=X2#
4260  DY#=D2#
```

Listing 1. BASIC Program for Weibull Parameter Estimation (Continued)

```
4270  NC=NC+1
4280  RETURN
4290  JK=2: NC=2: RETURN
4300  PRINT"FAILED TO CONVERGE"
4310  JK=3: NC=1: RETURN
4320  JK=4: RETURN
5000  FOR J=1 TO N: IZ(J)=J: NEXT J
5010  IF N=1 RETURN
5020  NM=N-1
5030  FOR K=1 TO N
5040  FOR J=1 TO NM
5050  N1=IZ(J)
5060  N2=IZ(J+1)
5070  IF V(N1) < V(N2) GOTO 5090
5080  IZ(J+1)=N1: IZ(J)=N2
5090  NEXT J
5100  NEXT K
5110  RETURN
```

```
      COMMON /BLOCK1/INT,TIME,MM,IHIST
      DIMENSION IS(100),RANKMD(100),A(100),ORDER(100)
      DIMENSION YX(100),INT(100),TIME(100),V(100)
      DIMENSION IZ(100),X(100)
      DATA ANO/'N'/,PERINT/50./
      IPRNT=1
      IPOP=0
      ISUM=0
      WRITE (6,4001)
 4001 FORMAT (2X,' ARE YOU INPUTTING A HISTOGRAM OF SUSPENSIONS?',
     ?/,2X,'ANSWER Y OR N')
      READ (5,4002) ANS
 4002 FORMAT (A1)
      IF (ANS .EQ. ANO) GOTO 701
      IPOP=1
      WRITE (6,4003)
 4003 FORMAT (2X,'AN INTERVAL SIZE OF 50 IS ASSUMED,',
     ?/,2X,'O.K.?....ANSWER Y OR N')
      READ (5,4002) ANS
      IF(ANS .NE. ANO) GOTO 8801
      WRITE (6,4004)
 4004 FORMAT (2X,'PLACE THE INTERVAL SIZE YOU WILL USE IN',
     ?/,2X,'CC 1 - 10 W/DECIMAL')
      READ (5,4005) PERINT
 4005 FORMAT (F10.0)
 8801 CONTINUE
      WRITE (6,4006)
 4006 FORMAT (2X,'PLACE THE NUMBER OF ELEMENTS IN EACH'
     ?,/,2X,'INTERVAL OF THE HISTOGRAM IN CC 1 - 10,',/,2X,
     ?'W/DECIMAL . . USE -99. TO INDICATE THE END')
      M=0
  211 M=M+1
      READ (5,1007)XINT
 1007 FORMAT (F10.0)
      IF (XINT .EQ. -99.) GOTO 212
      INT(M) = XINT
      GOTO 211
  212 MM = M - 1
      WRITE (6,3090)(INT(KL),KL=1,MM)
 3090 FORMAT (10I4)
      ISUM = 0
      DO 2001 J=1,MM
      TIME(J)=PERINT/2.+(J-1)*PERINT
      ISUM=ISUM+INT(J)
 2001 CONTINUE
C INTERMEDIATE PRINT
      WRITE(6,798)(TIME(J),J=1,MM)
  798 FORMAT(8F10.1)
C INTERMEDIATE PRINT
  701 WRITE(6,2000)
```

Listing 2. FORTRAN Program for Weibull Parameter Estimation (Continued)

```
2000  FORMAT (2X,'INPUT THE FAILURE DATA AND SUSPENSIONS'
     ?,/,2X,'IN COLS 1-10 WITH DECIMAL, -99999.'
     ?,/,2X,'INDICATES THE END OF DATA (NEGATIVES'
     ?,/,2X,'INDICATE SUSPENSIONS)..UNLESS HISTOGRAM'
     ?,/,2X,'WAS INPUT')
      I=0
   1  I=I+1
      READ (5,101) A(I)
 101  FORMAT(F10.0)
      IF (A(I) .EQ. 0.0)GOTO 3
      IF (A(I) .EQ. -99999.)GOTO 2
      GOTO 1
   3  I=I-1
      GOTO 1
   2  N=I-1
      BN=N+ISUM
      DO 4 J=1,N
   4  V(J)=ABS(A(J))
      CALL ORD(V,N,IZ)
      DO 22 I=1,N
      ISUB=IZ(I)
  22  X(I)=A(ISUB)
      DO 23 I=1,N
  23  A(I)=X(I)
      BN1=BN+1
      DJ=1.0
      BJ=0.0
      M=0
      SUMX=0.0
      SUMY=0.0
      SUMXX=0.0
      SUMYY=0.0
      SUMXY=0.0
      WRITE (6,990)
 990  FORMAT (2X,'PT.',4X,'DATA',4X,'ORDER',4X, 'MEDIAN RANK')
      DO 630 K=1,N
      ISUM=0
      IF (IPOP .EQ. 0)GOTO 632
      DO 631 J=1,MM
      IF (TIME(J) .LT. A(K))ISUM=ISUM+INT(J)
 631  CONTINUE
      IS(K)=ISUM
 632  BK=ISUM+K
      IF (IPOP .EQ. 1)BK=BK-1.0
      IF(IPOP .EQ. 1 .AND. K .EQ. 1)DJ=(BN1-BJ)/(BN1-BK)
      IF(K .EQ. 1) GOTO 3911
      IF (IPOP .EQ. 0) GO TO 3911
      IF(IS(K) .EQ. IS(K-1))GOTO 3911
3901  DJ=(BN1-BJ)/(BN1-BK)
3911  IF (A(K))390,900,400
 390  DJ=(BN1-BJ)/(BN1-BK)
      GOTO 630
```

```
400   BJ=BJ+DJ
      ORDER(K)=BJ
      RANKMD(K)=(ORDER(K)-.3)/(BN+.4)
      XXX=ALOG(A(K))
      YPRIME=1./(1.-RANKMD(K))
      Y=ALOG(ALOG(YPRIME))
      YX(K)=Y
      WRITE(6,300)K,A(K),ORDER(K),RANKMD(K)
300   FORMAT (I6,F10.1,F10.4,F12.5)
      M=M+1
      SUMX=SUMX+XXX
      SUMXX = SUMXX + XXX * XXX
      SUMY=SUMY+Y
      SUMYY=SUMYY + Y * Y
      SUMXY=SUMXY + XXX * Y
630   CONTINUE
C WRITE (6,800)SUMX,SUMY,SUMXX,SUMYY,SUMXY
800   FORMAT (2X,'SUMX=',E20.7,'SUMY=',E20.7,/,2X,
     ?'SUMXX=',E20.7,'SUMYY=',E20.7,'SUMXY=',E20.7)
      GM=M
      BETA=(GM*SUMYY-SUMY*SUMY)/(GM*SUMXY-SUMX*SUMY)
      ALPLN=(BETA*SUMX-SUMY)/GM
      AVED=ALPLN/BETA
      ETA=EXP(AVED)
      WRITE(6,3101)
3101  FORMAT(2X,'THE FOLLOWING ESTIMATES ARE RANKED REGRESSION',
     ?'ESTIMATES')
      WRITE (6,3100)BETA,ETA
3100  FORMAT (/,2X,'BETA=',F10.4,' ETA=',E20.7)
      R=0.0
      IF((SUMXX-SUMX*SUMX/GM) .LT. 0.0)GOTO 7871
      XNUM=SUMXY-SUMX*SUMY/GM
      DENOM=SQRT((SUMXX-SUMX*SUMX/GM)*(SUMYY-SUMY*SUMY/GM))
      R=XNUM/DENOM
      IF (R .GT. 1.0)R=1.0
7871  RSQ=R*R
      WRITE(6,3200)R,RSQ
3200  FORMAT (2X,'R=',F10.5,'R**2=',F10.5)
      WRITE (6,5001)
5001  FORMAT (2X,' DO YOU WISH TO DO MAXIMUM LIKELIHOOD ESTIMATION?',
     ?/,2X,'ANSWER Y OR N')
      READ (5,4002) ANS
      IF (ANS .EQ. ANO) GOTO 900
      IHIST=IPOP
      CALL MAXL(A,N,IPRNT,BML,TML,BETA)
900   CONTINUE
      STOP
      END
      SUBROUTINE ORD(A,N,IZ)
      DIMENSION A(1),IZ(1)
      DO 1 J=1,N
```

Listing 2. FORTRAN Program for Weibull Parameter Estimation (Continued)

```
1   IZ(J)=J
    IF (N .EQ. 1) RETURN
    NM = N - 1
    DO 2 K=1,N
    DO 2 J=1,NM
    N1=IZ(J)
    N2=IZ(J+1)
    IF (A(N1) .LT. A(N2)) GOTO 2
    IZ(J+1)=N1
    IZ(J)=N2
2   CONTINUE
    RETURN
    END
    SUBROUTINE MAXL(T,NUM,IPRNT,BML,TML,BETA)
    DIMENSION TT(100),T(1),XX(100),INT(100),TIME(100)
    COMMON /BLOCK/TT,NSAMP
    COMMON /BLOCK1/INT,TIME,MM,IHIST
    COMMON /BLOCK2/XX,NFAIL
    IH=IHIST
    NSAMP=NUM
    NFAIL=0
    DO 1 I=1,NSAMP
    TT(I)=ABS(T(I))
    IF(T(I) .LT. 0.0D0)GO TO 1
    NFAIL=NFAIL+1
    XX(NFAIL)=T(I)
1   CONTINUE
    TOL=.000001
    NLIM=100
    X=BETA
    PB=0.001
    NCT=0
    DELX=.001
    DELY=.01
30  PRN= AUX(X)
    IF (IPRNT .EQ. 1)WRITE(6,206)X,PRN
206 FORMAT(2X,' BETA=',E15.5,' LN MAXLIKELIHOOD=',E20.7)
    CALL SLOPE(X,PB,PRN,DELX,DELY,TOL,ISIG,NCT,NLIM,1)
    GO TO (30,50,40,50),ISIG
    GO TO 30
40  IF (IPRNT .EQ. 1)WRITE(6,205)FPR,PB,PRN
205 FORMAT(2X,' ITERATION FAILURE ',3E20.7)
    IFLAG=1
    RETURN
50  BML=X
    SUM=0.0D0
    RN=NFAIL
    DO 110 I=1,NUM
110 SUM=SUM+TT(I)**BML
    IF (IHIST .EQ. 0) GO TO 112
    DO 111 I=1,MM
111 SUM=SUM+FLOAT(INT(I))*TIME(I)**BML
```

Listing 2. FORTRAN Program for Weibull Parameter Estimation (Continued)

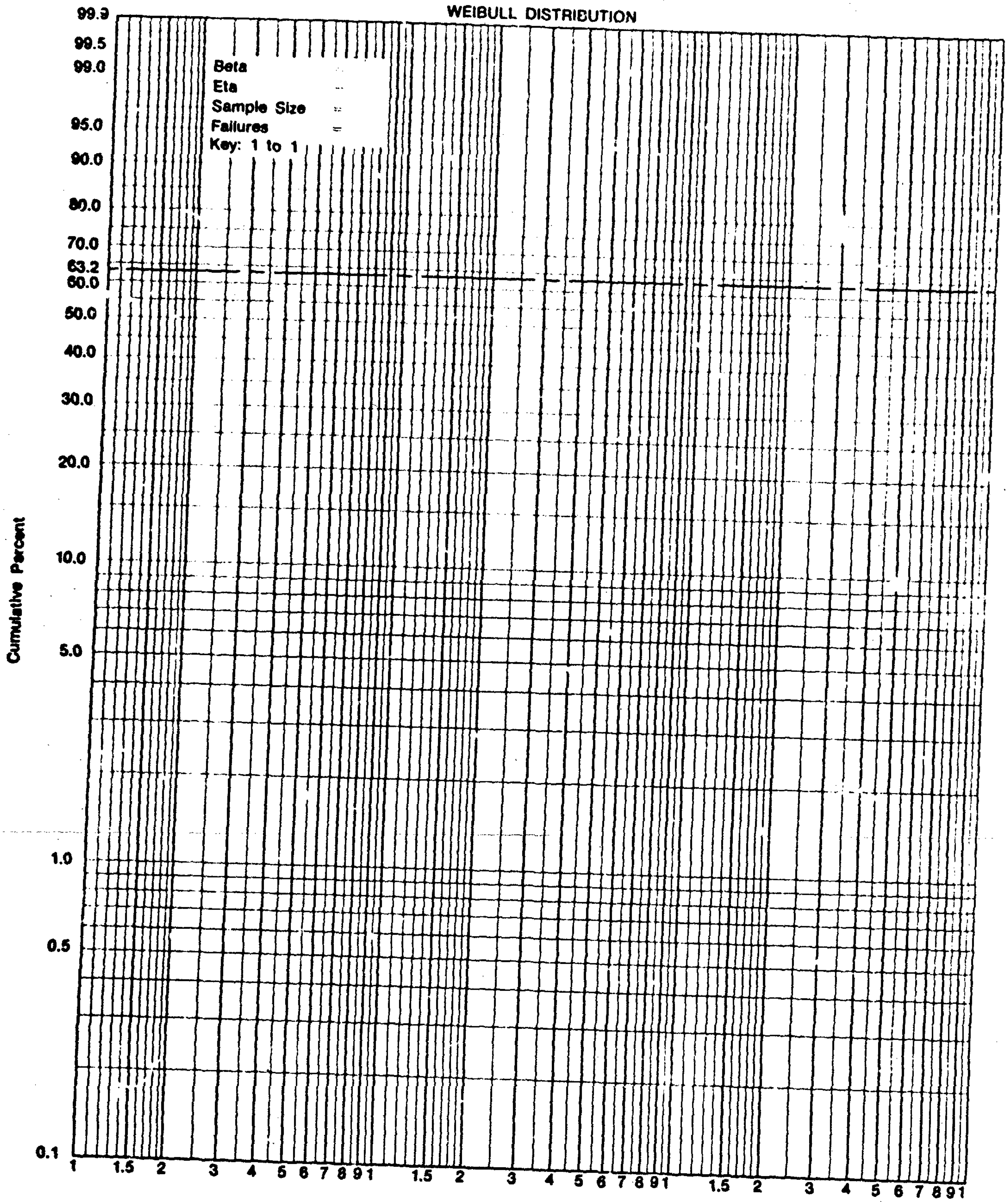
```
112  SUM=SUM/RN
     TML=SUM**(1.0/BML)
     WRITE(6,996)
996  FORMAT(2X,' MAXIMUM LIKELIHOOD ESTIMATES FOR THIS CASE FOLLOW')
     WRITE(6,995)BML,TML
995  FORMAT(2X,' BETA=',F10.3,' ETA=',F20.2)
     RETURN
     END
     FUNCTION AUX(X)
     DIMENSION T(100),XX(100),INT(100),TIME(100)
     COMMON /BLOCK/T,N
     COMMON /BLOCK1/INT,TIME,MM,IHIST
     COMMON /BLOCK2/XX,NFAIL
     ZZZ=NFAIL
     SUM1=0.0
     SUM2=0.0
     SUM3=0.0
     IF(ABS(X) .GT. 15.0 .OR. X .LE. 0.0)X=0.1
     DO 10 I=1,N
     SUM1=SUM1+T(I)**X
10   SUM2=SUM2+ALOG(T(I))*T(I)**X
     IF (IHIST .EQ. 0)GO TO 11
     DO 20 I=1,MM
     SUM1=SUM1+FLOAT(INT(I))*TIME(I)**X
20   SUM2=SUM2+FLOAT(INT(I))*ALOG(TIME(I))*TIME(I)**X
11   DO 15 I=1,NFAIL
15   SUM3=SUM3+ALOG(XX(I))
     AUX=SUM2/SUM1-SUM3/ZZZ-1.0/X
     RETURN
     END
     SUBROUTINE SLOPE( X , YA , YB ,X1,DEL1,TOL,JK,NCT,NTIME,LOOP)
     JK=1
     IF((ABS((YA-YB)/YA))-TOL)6,6,3
  3  IF(NCT-1)1,1,2
  1  X1=X
     DEL1=YA-YB
     NCT=NCT+1
     X=X*1.02
     GO TO 9
  2  IF(NCT-NTIME)5,5,4
  5  X2=X
     DEL2=YA-YB
     IF (ABS(DEL2-DEL1) .LT. 1.E-06) GO TO 20
     X=X2-DEL2*(X2-X1)/(DEL2-DEL1)
     IF(X)8,8,10
10   IF(X-X2)11,7,12
11   IF((X/X2)-.6)13,13,8
13   X=X2*.75
     GO TO 8
12   IF((X/X2)-1.4)8,14,14
```

Listing 2. FORTRAN Program for Weibull Parameter Estimation (Continued)

```
14   X=X2*1.25
     GO TO 8
   7 X=X2*1.02
   8 X1=X2
     DEL1=DEL2
     NCT=NCT+1
     GO TO 9
   6 JK=2
     NCT=2
     GO TO 9
   4 WRITE(6,100)LOOP,X,YA,YB
100  FORMAT(1H0,'CONVERGENCE FAILURE IN LOOP',I2/1H ,'X =',E14.8,4X,' YA=',
    ?E14 8,4X,'YB =',E14.8 /)
     JK=3
     NCT=1
   9 CONTINUE
     RETURN
20   JK=4
     RETURN
     END
```

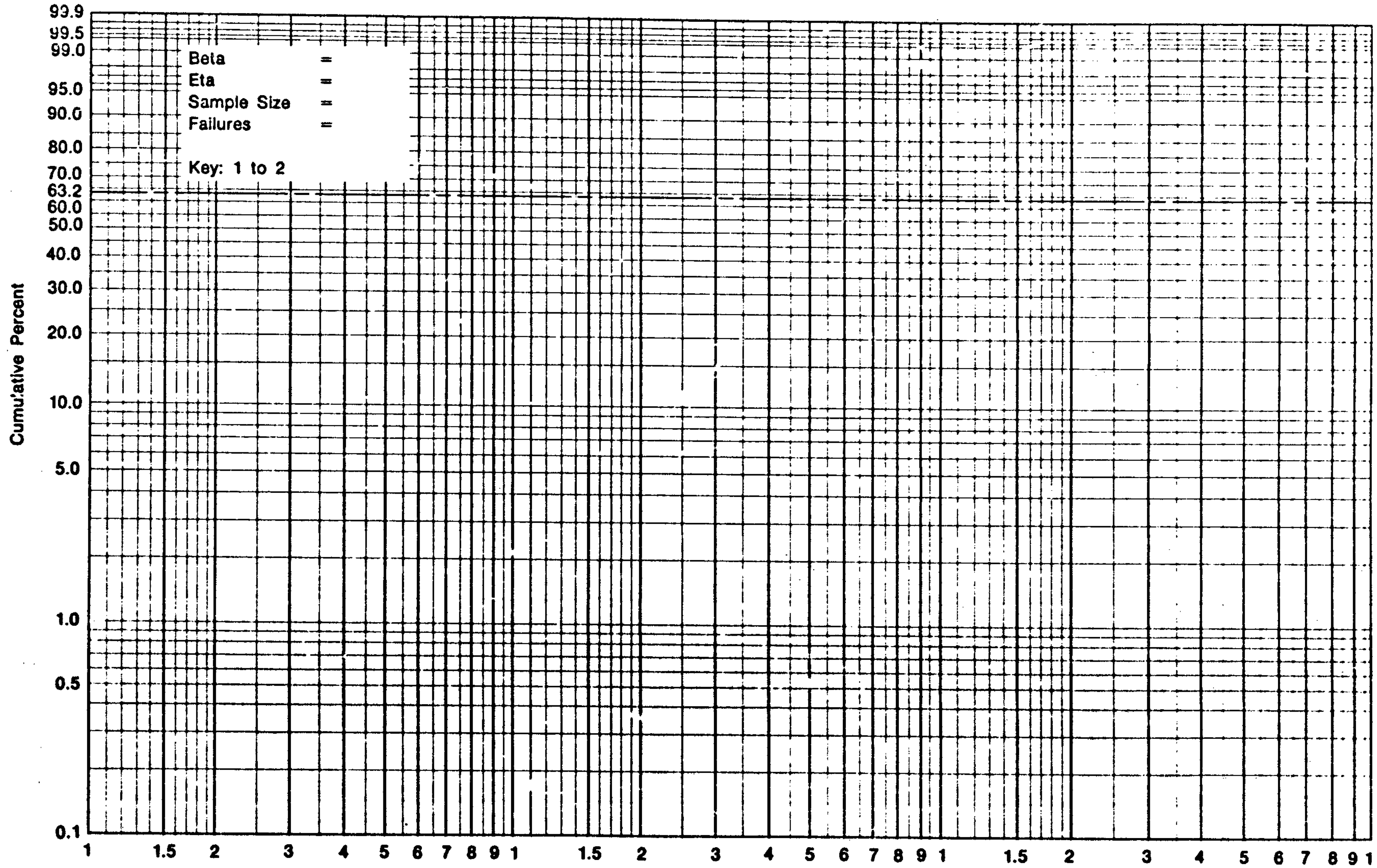
APPENDIX I

WEIBULL GRAPHS


WEIBULL DISTRIBUTION
Beta
Eta
Sample Size
Failures
Key: 1 to 1
Cumulative Percent
99.9
99.5
99.0
95.0
90.0
80.0
70.0
63.2
60.0
50.0
40.0
30.0
20.0
10.0
5.0
1.0
0.5
0.1
1 1.5 2 3 4 5 6 7 8 9 1 1.5 2 3 4 5 6 7 8 9 1 1.5 2 3 4 5 6 7 8 9 1

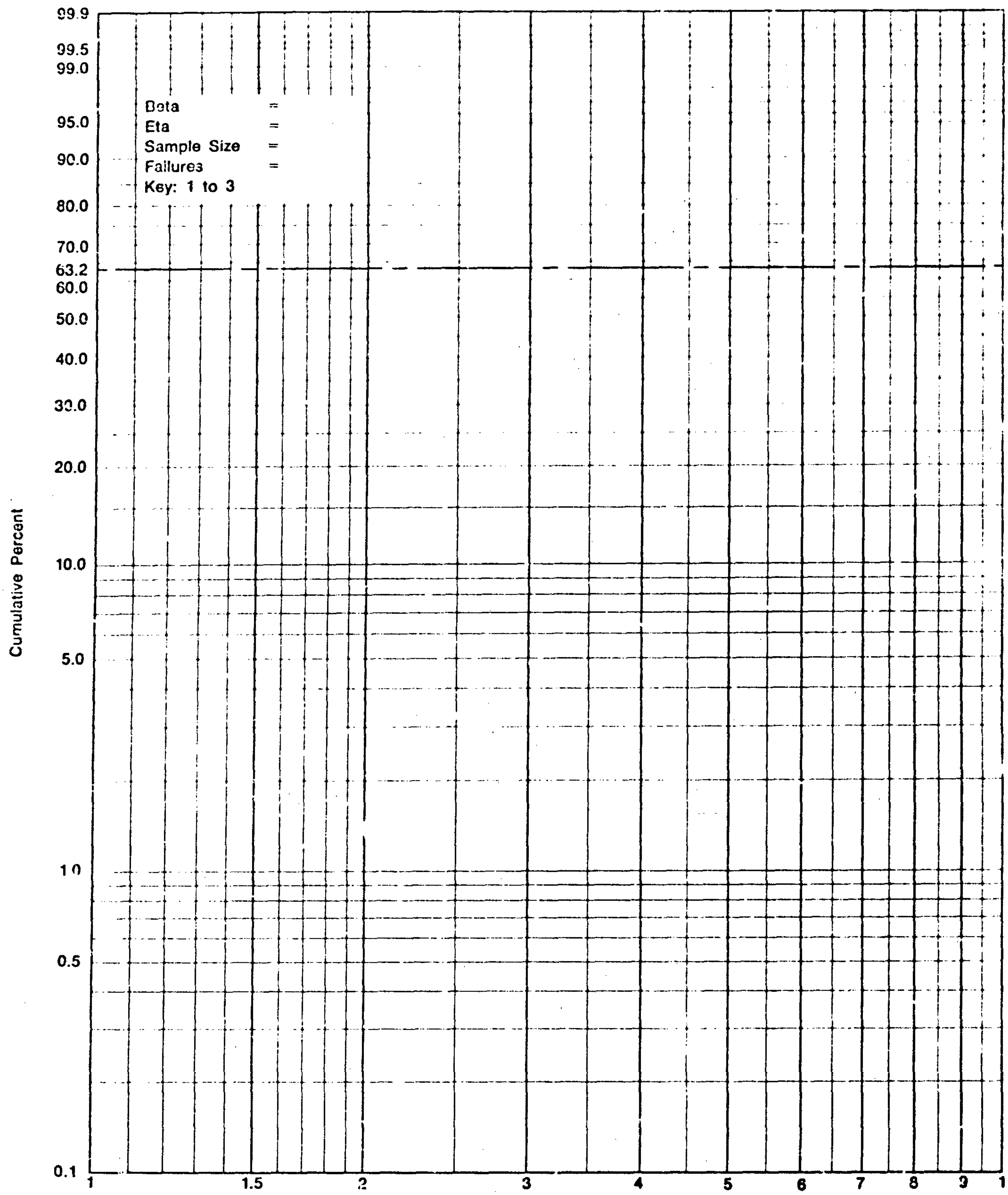
FD 265609

WEIBULL DISTRIBUTION



FD 265690

WEIBULL DISTRIBUTION



FD 265691

APPENDIX J

ANSWERS TO PROBLEMS

1. CHAPTER 1 ANSWERS: None

2. CHAPTER 2 ANSWERS

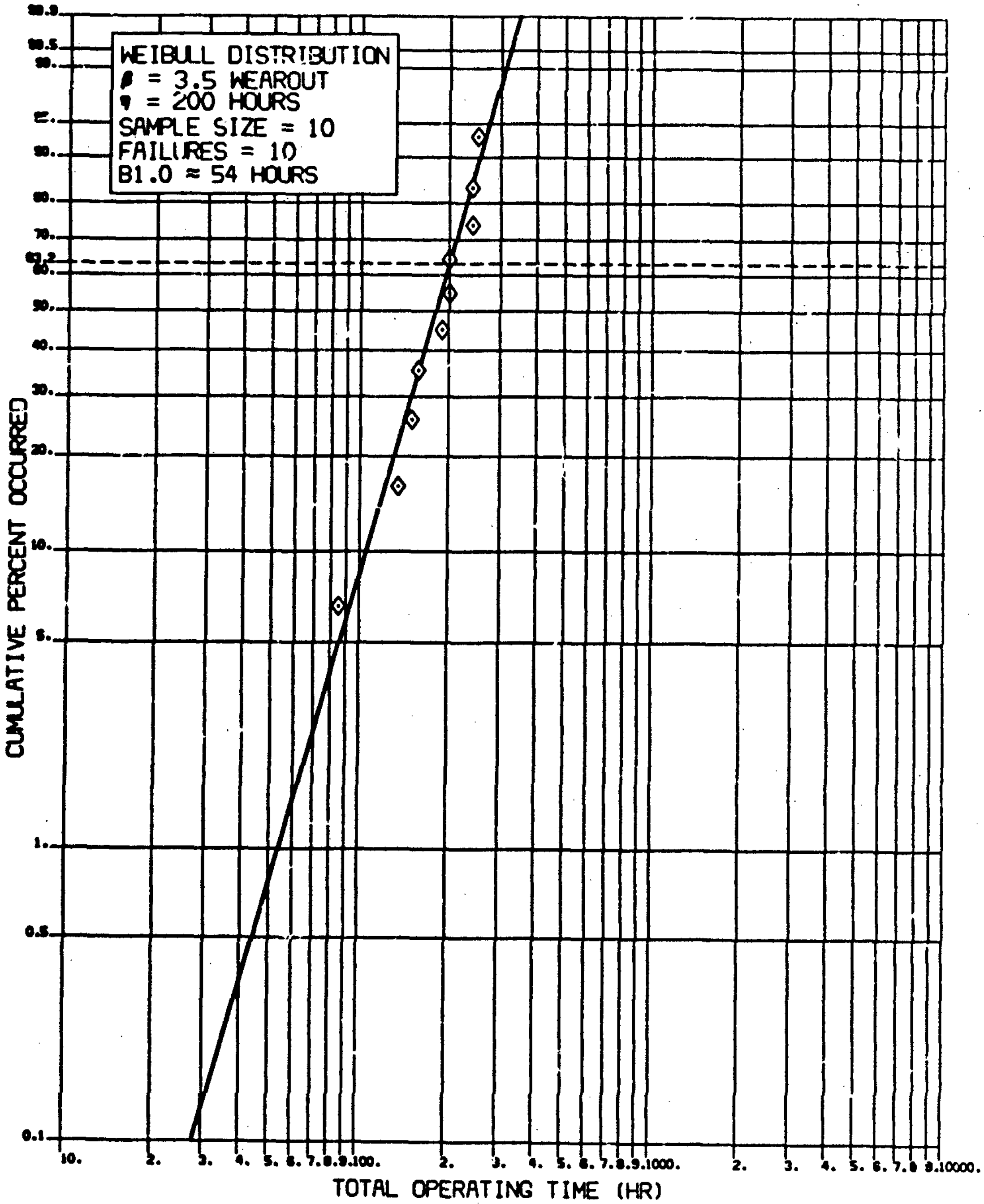
2.1 Problem 2-1

a. $\beta = 3.5$ $\eta = 200$ hours
b. Yes. Wearout or fatigue failures usually have steep slopes
c. $B_{1.0} = 54$ hours.

Rank	*Time (hr)*	*Median* Ranks*
1	85	6.6%
2	135	16.2
3	150	25.8
4	150	35.5
5	190	45.1
6	200	54.8
7	200	64.4
8	240	74.1
9	240	83.7
10	250	93.3

*From median rank table

See Figure J.1.



FD 271896

Figure J.1. Problem 2-1

2.2 Problem 2-2: Answers

a. $\beta = 0.75 \quad \eta = 29$ hours

b. Infant Mortality

c. Serial numbers are very close. A batch problem may be suspect.

Rank	*Comment*	*New Rank*	*Median Rank*
1	Failure	1	8.3%
2	Failure	2	20.2
3	Suspension		
4	Suspension		
5	Failure	3.4[1]	36.9
6	Suspension		
7	Failure	5.3[2]	59.5
8	Failure	7.2[3]	82.1

[1]Rank Increment $= \dfrac{(N + 1) - (\text{previous rank order number})}{1 + (\text{number of items beyond present suspended items})}$

$$= \frac{9 - 2}{1 + 4} = \frac{7}{5} = 1.4$$

$$= 2 + 1.4 = 3.4$$

[2]Rank Increment $= \dfrac{9 - 3.4}{1 + 2} = \dfrac{5.6}{3} = 1.87$

New Rank $= 3.4 + 1.87 = 5.27 \approx 5.3$

[3]New Rank $= 5.3 + 1.87 = 7.17 \approx 7.2$

See Figure J.2.

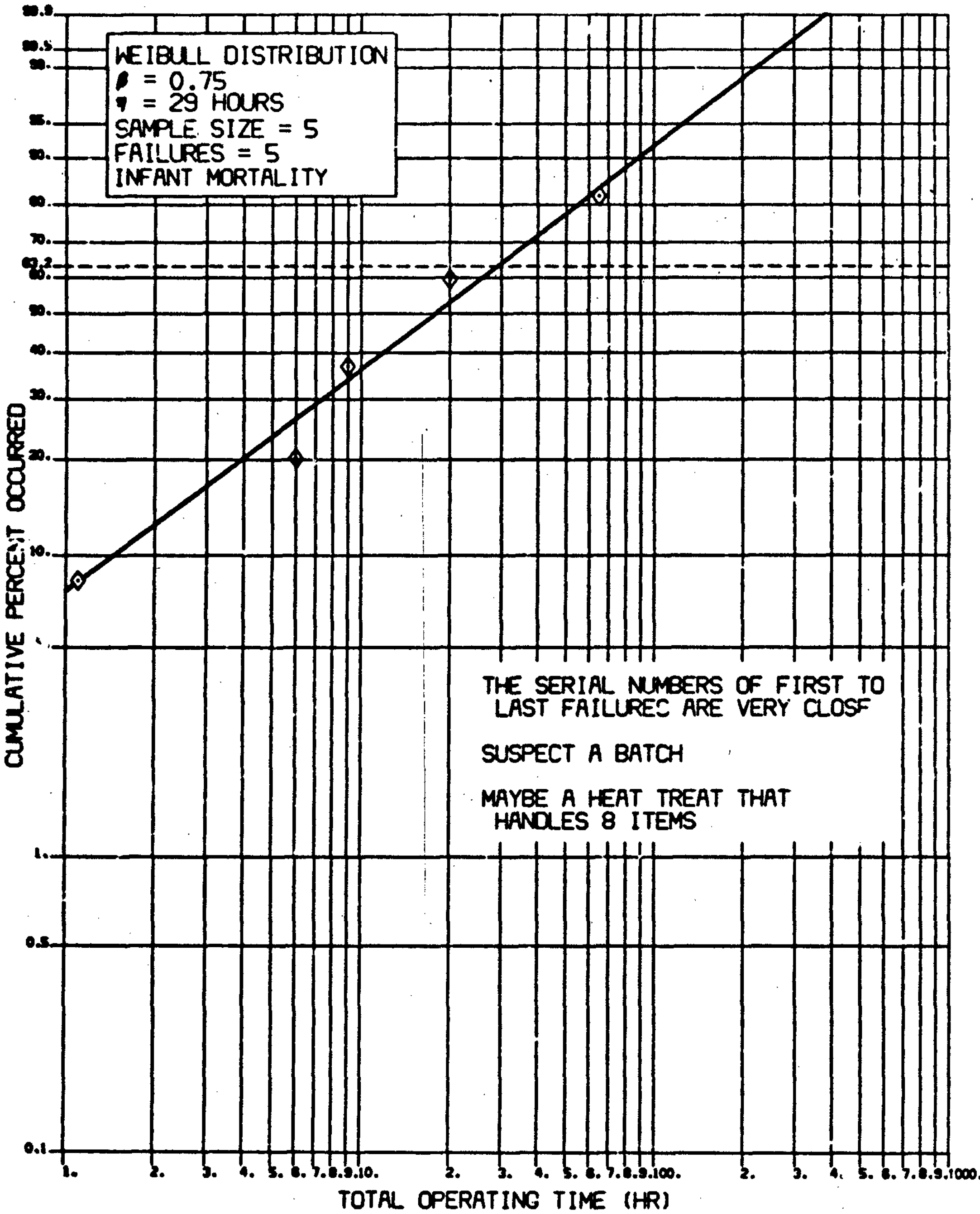


Figure J.2. Problem 2-2, Infant Mortality

2.3 Problem 2-3: Answers

Rank	*Time*	*Median* Ranks*
1	90 hr	8.3
2	130	20.1
3	165	32.0
4	220	44.0
5	275	55.9
6	370	67.9
7	525	79.8
8	1,200	91.7

*From median rank table

Using the formula:

$$t_o = t_2 - \frac{(t_3 - t_2)\,(t_2 - t_1)}{(t_3 - t_2) - (t_2 - t_1)}$$

where $t_1 = 90$, $t_2 = 185$, $t_3 = 1{,}200$

$$t_o = 185 - \frac{(1{,}200 - 185)\,(185 - 90)}{(1{,}200 - 185) - (185 - 90)}$$

$$= 185 - \frac{(1{,}015)\,(95)}{1{,}015 - 95}$$

$$= 185 - \frac{96{,}425}{920}$$

$$= 185 - 104.8$$

$$t_o = 80.2$$

Time	*Time — 80.2 hr*	*Median* Rank*
90 hr	9.8	8.3
130	49.8	20.1
165	84.8	32.0
220	139.8	44.0
275	194.8	55.9
370	289.8	67.9
525	444.8	79.8
1,200	1,119.8	91.7

See Figures J.3 and J.4.

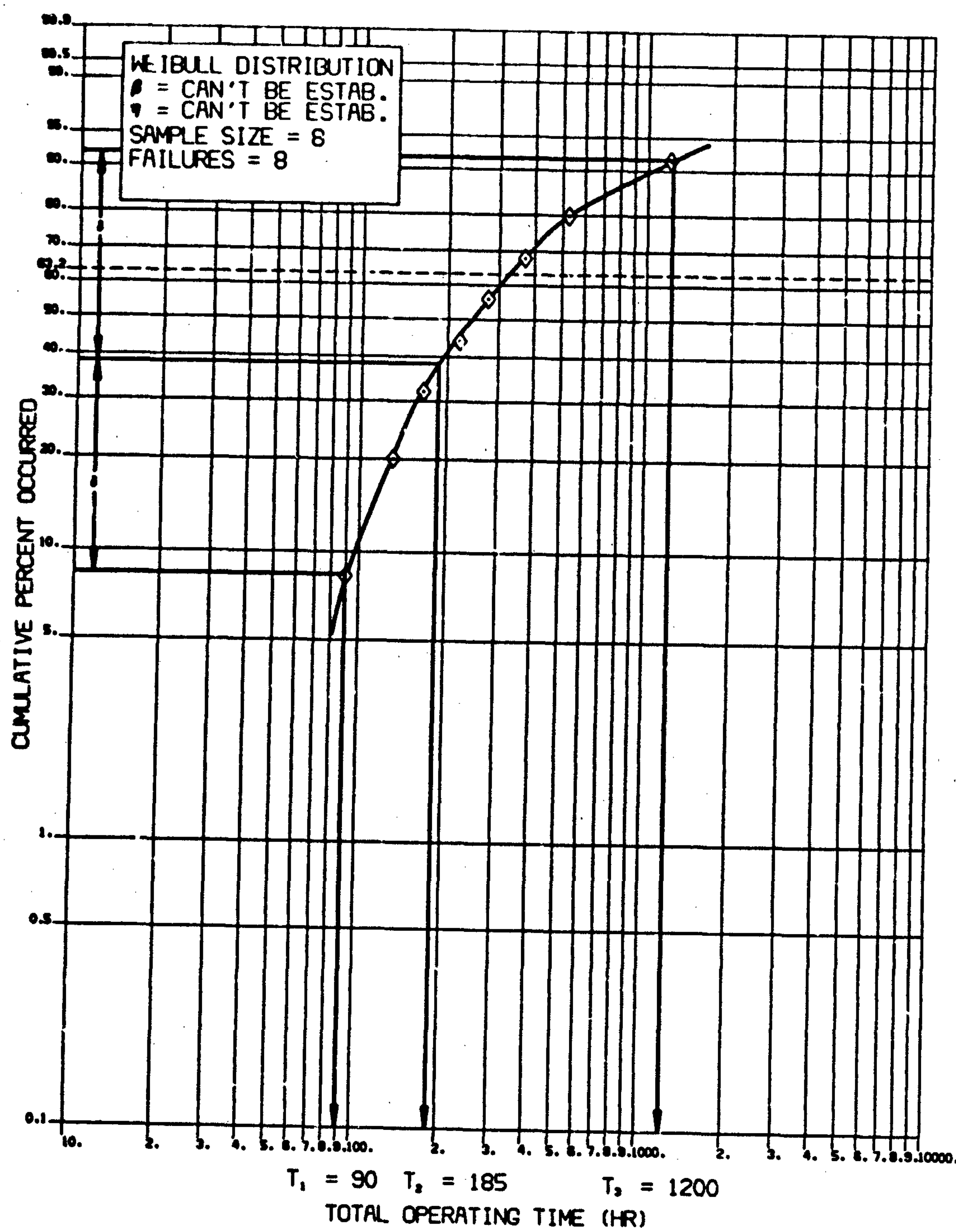


Figure J.3. Problem 2-3, Curved Weibull

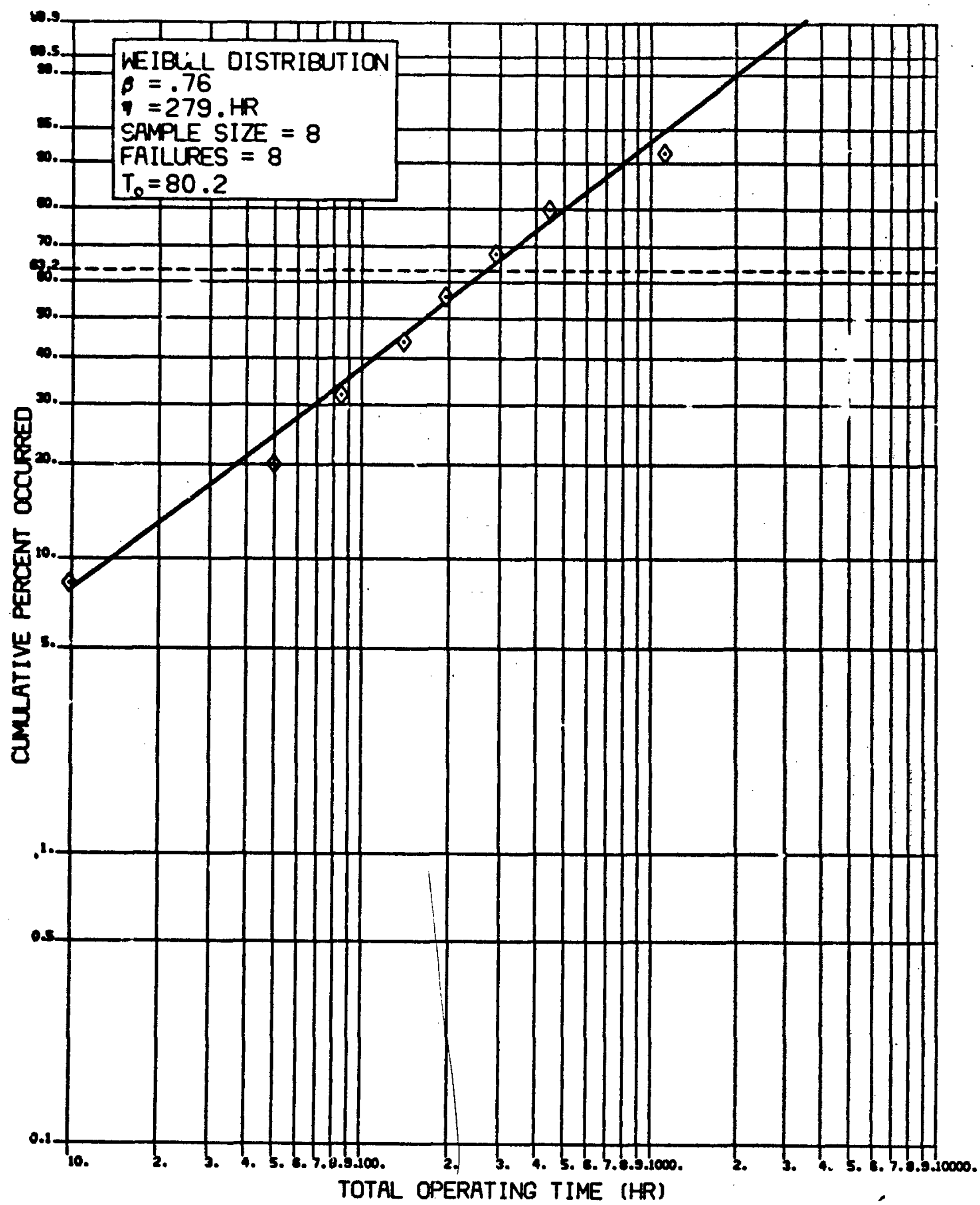


Figure J.4. Problem 2-3, Overall Population

3. CHAPTER 3 ANSWERS

3.1 Problem 3-1

(a) Today

Number Engines	*Time on Each Engine*	*F(t)*	*F(t)·N*
20	150	0.0033	0.067
20	200	0.008	0.16
20	250	0.0155	0.31
20	300	0.0266	0.533
20	350	0.042	0.84
			Σ = 1.909

(b) In Six Months

Number Engines	*Time on Each Engine*	*F(t)*	*F(t)·N*
20	300	0.0266	0.533
20	350	0.042	0.84
20	400	0.062	1.24
20	450	0.087	1.74
20	500	0.118	2.35
			Σ = 6.704

Therefore, additional failures = 6.704 − 1.909 = 4.8

3.2 Problem 3-2: Turbine Airfoil Unscheduled Engine Removals

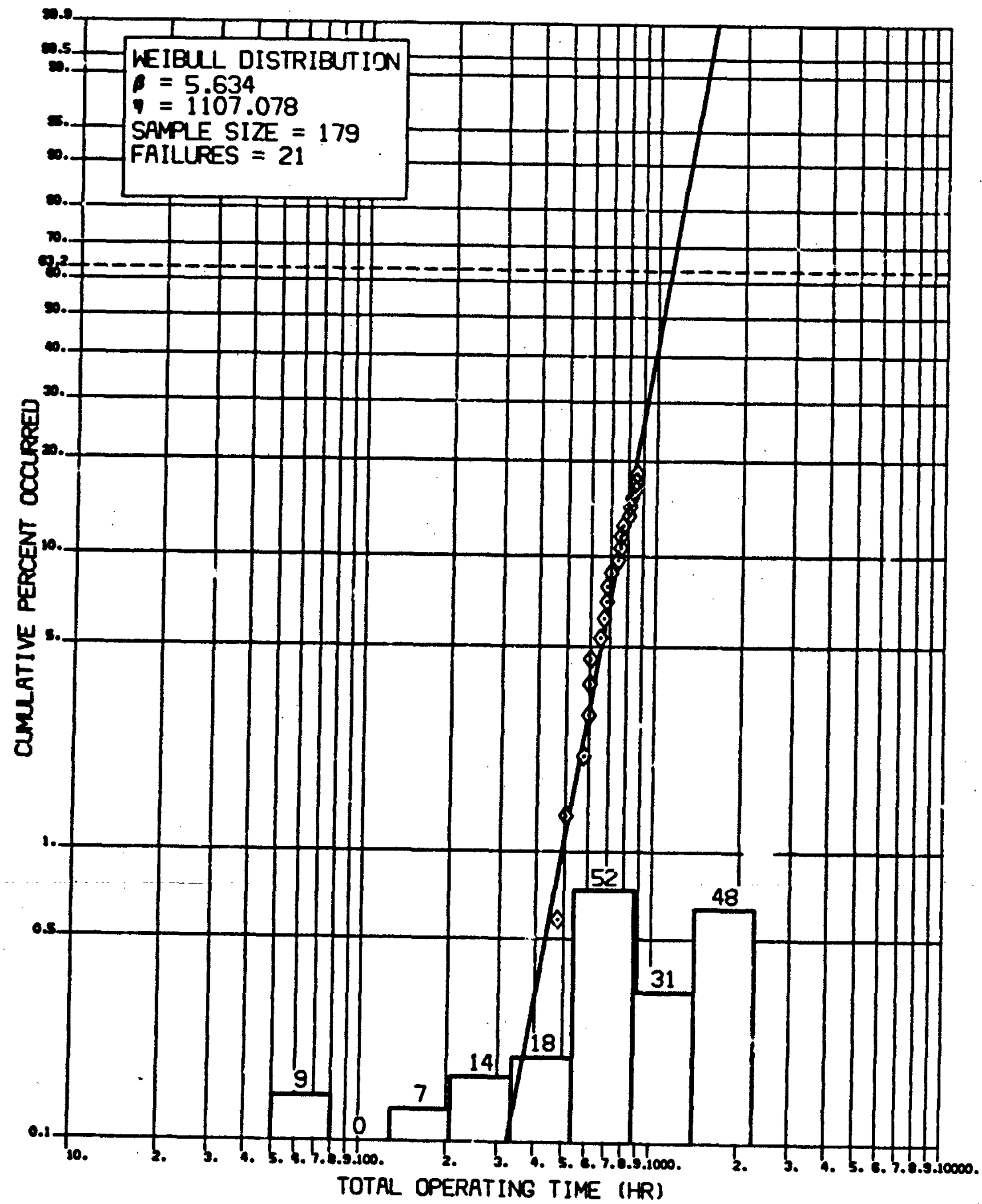
(a) First, overall using Figure 3.13 population:

Point	*Data*	*Mean Order*	*Median Rank*
47	476.000	1.343	0.00582
48	504.000	2.687	0.01330
58	576.000	4.128	0.02134
59	596.000	5.570	0.02937
60	600.000	7.011	0.03741
61	603.000	8.453	0.04545
62	649.000	9.894	0.05348
72	667.000	11.455	0.06218
73	681.000	13.016	0.07088
74	684.000	14.576	0.07958
75	701.000	16.137	0.08828
76	741.000	17.697	0.09698
85	755.000	19.388	0.10640
86	756.000	21.079	0.11582
87	770.000	22.769	0.12525
88	806.000	24.460	0.13467
89	812.000	26.151	0.14409
90	821.000	27.841	0.15352
91	845.000	29.532	0.16294
93	850.000	31.242	0.17247
100	855.000	33 078	0.18271

then, using the Figure 3.14 population

Point	*Data*	*Mean Order*	*Median Rank*
3	584.000	1.067	0.02442
4	701.000	2.133	0.05839
6	770.000	3.240	0.09361
7	812.000	4.346	0.12884
8	821.000	5.452	0.16407
9	845.000	6.558	0.19930

The associated Weibull Plots are in Figures J.5 and J.6; they "seem" different. However, Figure J.7 illustrates the total population Weibull with confidence bounds (from Chapter 7); since the Location A Weibull lies outside the All Locations Weibull, the two Weibulls are significantly different.



FD 272252

Figure J.5. Problem 3-2, Overall Population

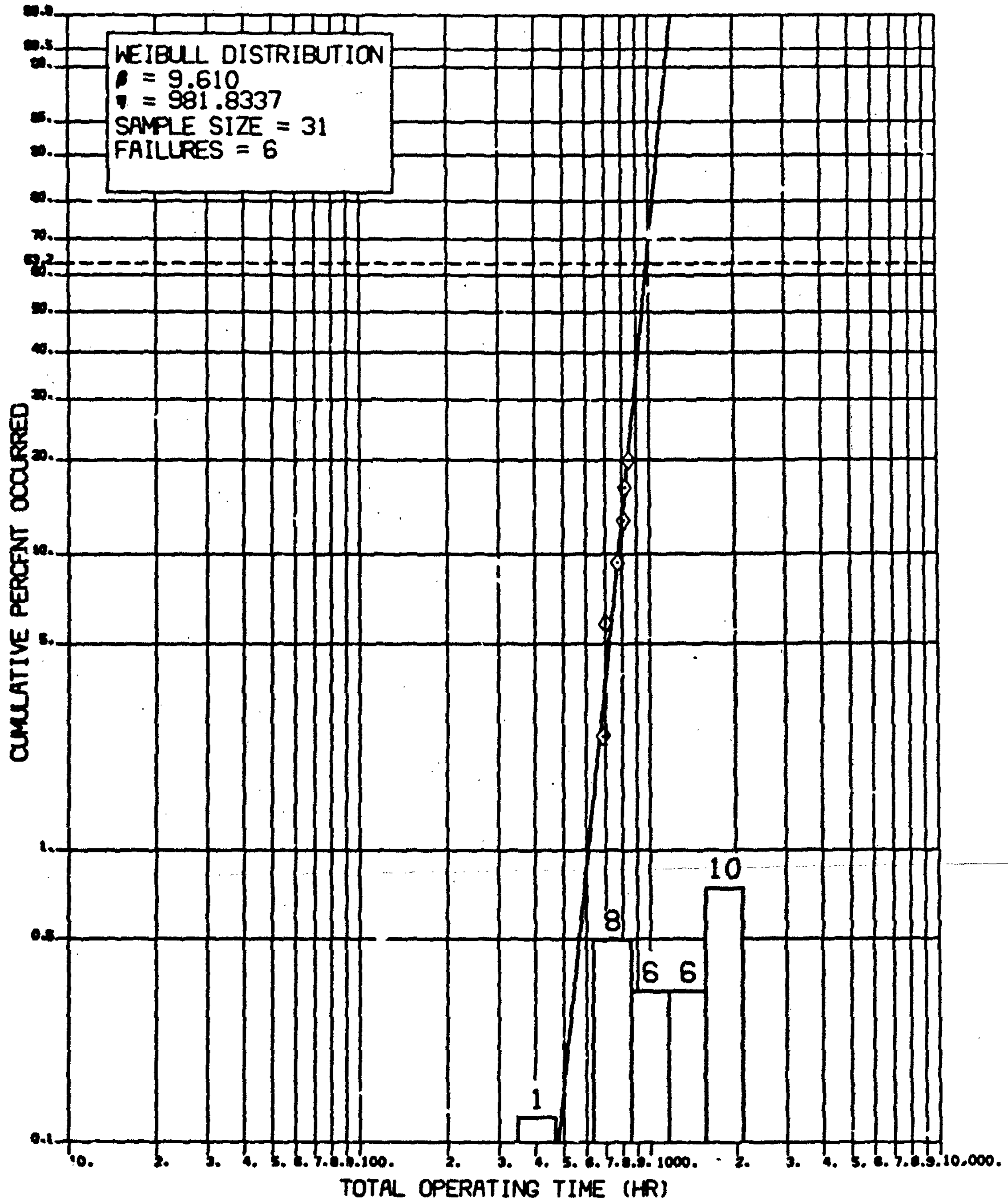


Figure J.6. Problem 3-2, Location A Only

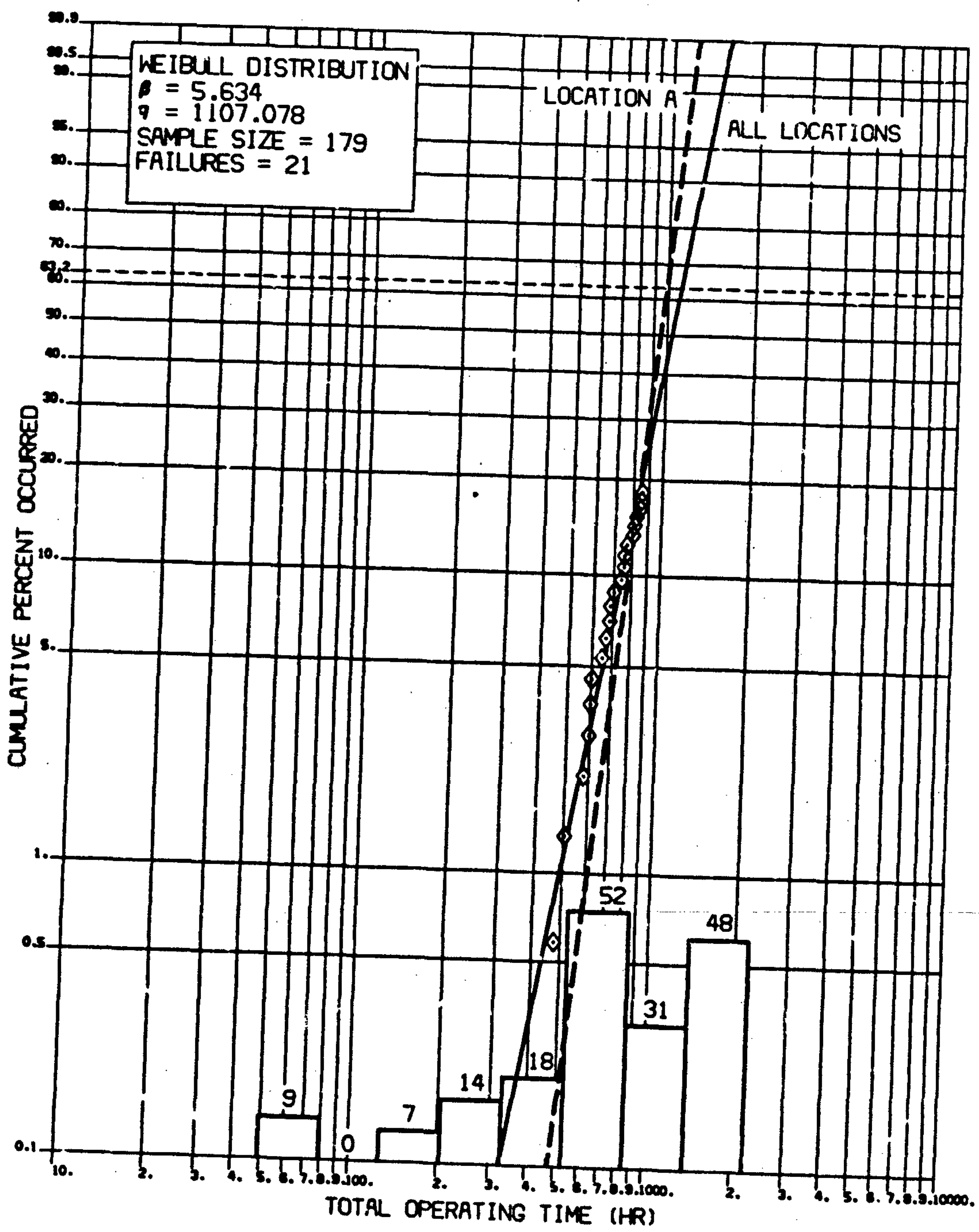


Figure J.7. Problem 3-2

FD 2722254

(b) For the entire population, in 12 months

Number of Units (N)	*Cumulative Units*	*t*	*t+360*	*F(t)*	*F(t+360)*	$G(t) = \frac{F(t+360)-F(t)}{1-F(t)}$	$G(t) \cdot N$
9.0	9.0	50.0	410.0	0.0	0.0	0.0	
7.0	16.0	150.0	510.0	0.0	0.0	0.0	0.1
14.0	30.0	250.0	610.0	0.0	0.0	0.0	0.5
9.0	39.0	350.0	710.0	0.0	0.1	0.1	0.7
7.0	46.0	450.0	810.0	0.0	0.2	0.2	1.1
9.0	55.0	550.0	910.0	0.0	0.3	0.3	2.4
9.0	64.0	650.0	1,010.0	0.0	0.4	0.4	3.8
8.0	72.0	750.0	1,110.0	0.1	0.6	0.6	4.8
7.0	79.0	850.0	1,210.0	0.2	0.8	0.8	5.3
6.0	85.0	950.0	1,310.0	0.3	0.9	0.9	5.3
6.0	91.0	1,050.0	1,410.0	0.5	1.0	1.0	5.7
6.0	97.0	1,150.0	1,510.0	0.7	1.0	1.0	5.9
7.0	104.0	1,250.0	1,610.0	0.9	1.0	1.0	7.0
6.0	110.0	1,350.0	1,710.0	1.0	1.0	1.0	6.0
7.0	117.0	1,450.0	1,810.0	1.0	1.0	1.0	7.0
8.0	125.0	1,550.0	1,910.0	1.0	1.0	1.0	8.0
7.0	132.0	1,650.0	2,010.0	1.0	1.0	1.0	7.0
8.0	140.0	1,750.0	2,110.0	1.0	1.0	1.0	8.0
8.0	148.0	1,850.0	2,110.0	1.0	1.0	1.0	8.0
4.0	152.0	1,950.0	2,310.0	1.0	1.0	1.0	4.0
3.0	155.0	2,050.0	2,410.0	1.0	1.0	1.0	3.0
2.0	157.0	2,150.0	2,510.0	1.0	1.0	1.0	2.0
1.0	158.0	2,250.0	2,610.0	1.0	1.0	1.0	1.0
							$\Sigma = 96.6$

For the entire population, in 24 months

Number of Units (N)	Cumulative Units	t	$t+720$	$F(t)$	$F(t+720)$	$G(t) = \dfrac{F(t+720) - F(t)}{1 - F(t)}$	$G(t) \cdot N$
9.0	9.0	50.0	770.0	0.0	0.1	0.1	1.1
7.0	16.0	150.0	870.0	0.0	0.2	0.2	1.6
14.0	30.0	250.0	970.0	0.0	0.4	0.4	5.3
9.0	39.0	350.0	1,070.0	0.0	0.6	0.6	5.1
7.0	46.0	450.0	1,170.0	0.0	0.7	0.7	5.2
9.0	55.0	550.0	1,270.0	0.0	0.9	0.9	7.9
9.0	64.0	650.0	1,370.0	0.0	1.0	1.0	8.7
8.0	72.0	750.0	1,470.0	0.1	1.0	1.0	7.9
7.0	79.0	850.0	1,570.0	0.2	1.0	1.0	7.0
6.0	85.0	950.0	1,670.0	0.3	1.0	1.0	6.0
6.0	91.0	1,050.0	1,770.0	0.5	1.0	1.0	6.0
6.0	97.0	1,150.0	1,870.0	0.7	1.0	1.0	6.0
7.0	104.0	1,250.0	1,970.0	0.9	1.0	1.0	7.0
6.0	110.0	1,350.0	2,070.0	1.0	1.0	1.0	6.0
7.0	117.0	1,450.0	2,170.0	1.0	1.0	1.0	7.0
8.0	125.0	1,550.0	2,270.0	1.0	1.0	1.0	8.0
7.0	132.0	1,650.0	2,370.0	1.0	1.0	1.0	7.0
8.0	140.0	1,750.0	2,470.0	1.0	1.0	1.0	8.0
8.0	148.0	1,850.0	2,570.0	1.0	1.0	1.0	8.0
4.0	152.0	1,950.0	2,670.0	1.0	1.0	1.0	4.0
3.0	155.0	2,050.0	2,770.0	1.0	1.0	1.0	3.0
2.0	157.0	2,150.0	2,870.0	1.0	1.0	1.0	2.0
1.0	158.0	2,250.0	2,970.0	1.0	1.0	1.0	1.0

$\Sigma = 128.8$

For the population at Location A, in 12 months:

Number of Units (N)	*Cumulative Units*	*t*	*t+360*	*F(t)*	*F(t+360)*	*G(t)*	*G(t).N*
0.0	0.0	50.0	410.0	0.0	0.0	0.0	0.0
0.0	0.0	150.0	510.0	0.0	0.0	0.0	0.0
0.0	0.0	250.0	610.0	0.0	0.0	0.0	0.0
1.0	1.0	350.0	710.0	0.0	0.0	0.0	0.0
0.0	1.0	450.0	810.0	0.0	0.1	0.1	0.0
0.0	1.0	550.0	910.0	0.0	0.4	0.4	0.0
1.0	2.0	650.0	1,010.0	0.0	0.7	0.7	0.7
1.0	3.0	750.0	1,110.0	0.1	1.0	1.0	1.0
2.0	5.0	850.0	1,210.0	0.2	1.0	1.0	2.0
2.0	7.0	950.0	1,310.0	0.5	1.0	1.0	2.0
2.0	9.0	1,050.0	1,410.0	0.9	1.0	1.0	2.0
2.0	11.0	1,150.0	1,510.0	1.0	1.0	1.0	2.0
2.0	13.0	1,250.0	1,610.0	1.0	1.0	1.0	2.0
1.0	14.0	1,350.0	1,710.0	1.0	1.0	1.0	1.0
1.0	15.0	1,450.0	1,810.0	1.0	1.0	1.0	1.0
2.0	17.0	1,550.0	1,910.0	1.0	1.0	1.0	2.0
1.0	18.0	1,650.0	2,010.0	1.0	1.0	1.0	1.0
1.0	19.0	1,750.0	2,110.0	1.0	1.0	1.0	1.0
2.0	21.0	1,850.0	2,210.0	1.0	1.0	1.0	2.0
2.0	23.0	1,950.0	2,310.0	1.0	1.0	1.0	2.0
2.0	25.0	2,050.0	2,410.0	1.0	1.0	1.0	2.0
							$\Sigma = 23.7$

For the population at Location A, in 24 months:

Number of Units (N)	*Cumulative Units*	*t*	*t+720*	*F(t)*	*F(t+720)*	*G(t)*	*G(t).N*
0.0	0.0	50.0	770.0	0.0	0.1	0.1	0.0
0.0	0.0	150.0	870.0	0.0	0.3	0.3	0.0
0.0	0.0	250.0	970.0	0.0	0.6	0.6	0.0
1.0	1.0	350.0	1,070.0	0.0	0.9	0.9	0.9
0.0	1.0	450.0	1,170.0	0.0	1.0	1.0	0.0
0.0	1.0	550.0	1,270.0	0.0	1.0	1.0	0.0
1.0	2.0	650.0	1,370.0	0.0	1.0	1.0	1.0
1.0	3.0	750.0	1,470.0	0.1	1.0	1.0	1.0
2.0	5.0	850.0	1,570.0	0.2	1.0	1.0	2.0
2.0	7.0	950.0	1,670.0	0.5	1.0	1.0	2.0
2.0	9.0	1,050.0	1,770.0	0.9	1.0	1.0	2.0
2.0	11.0	1,150.0	1,870.0	1.0	1.0	1.0	2.0
2.0	13.0	1,250.0	1,970.0	1.0	1.0	1.0	2.0
1.0	14.0	1,350.0	2,070.0	1.0	1.0	1.0	1.0
1.0	15.0	1,450.0	2,170.0	1.0	1.0	1.0	1.0
2.0	17.0	1,550.0	2,270.0	1.0	1.0	1.0	2.0
1.0	18.0	1,650.0	2,370.0	1.0	1.0	1.0	1.0
1.0	19.0	1,750.0	2,470.0	1.0	1.0	1.0	1.0
2.0	21.0	1,850.0	2,570.0	1.0	1.0	1.0	2.0
2.0	23.0	1,950.0	2,670.0	1.0	1.0	1.0	2.0
2.0	25.0	2,050.0	2,770.0	1.0	1.0	1.0	2.0
							Σ = 24.9

i.e., all of them will have failed and will have been fixed.

3.3 Problem 3-3

(A) $t = 1{,}000, F(t) = 1 - e^{(t/\eta)^{\beta}}$ equal $1 - e^{(t/19{,}735)^{1.26}}$

F(1,000) = 0.0231
Number of failures = (0.0231) (1,308) ≃ 30

(B) t = 2,000
F(2,000) = 0.0544
Number of failures = (0.0544) (1,308) ≃ 71

(C) t = 4,000
F(4,000) = 0.1253
Number of failures = (0.1253) (1,308) ≃ 164

(D) Inspection at 1,000 hours, makes units "good as new"
P (failure at 4,000 hours) = F(1,000) + F(1,000) + F(1,000) + F(1,000)
= 0.0231 + 0.0231 + 0.0231 + 0.0231
= 0.0924
Number of failures = (0.0924) (1,308) ≃ 121

(E) Inspection at 2,000 hours

P (failure at 4,000 hours) = F(2,000) + F(2,000)

0.0544 = 0.0544 + 0.1088

Number of failures = (0.1088) (1,308) $\simeq$ 142

3.4 Problem 3-4

Start with the top row of random numbers, and note that one's answer will vary depending on the random number string.

(1) Engine at 100 hours

$$F_A = 96{,}587\left(\ln\left(\frac{1}{1-0.329}\right)\right)^{1/.76} =$$

28,831.2 hours = 1,149 months from present

$$F_B = 4{,}996\left(\ln\left(\frac{1}{1-0.604}\right)\right)^{1/2.636} =$$

4,853.2hours = 190 months from present

$$F_C = 1{,}126\left(\ln\left(\frac{1}{1-0.615}\right)\right)^{1/7.4} =$$

1,118.9hours = 40 months from present

1,000 hour inspection is 900 hours or 36 months from present

Reset Mode A and C to "0"

$$F_A = 96{,}587\left(\ln\left(\frac{1}{1-0.3}\right)\right)^{1/.76} =$$

24,877 hours = 995 months from last inspection

$$F_C = 1{,}126\left(\ln\left(\frac{1}{1-0.07}\right)\right)^{1/7.4} =$$

789 hours = 31 months from last inspection

2,000 hour inspection is 40 months from last inspection

Past 48 months pt. of interest

(2) Engine at 200 hours

$$F_A = 96{,}587\left(\ln\left(\frac{1}{1-0.845}\right)\right)^{1/.76} =$$

219,214 hours = 8,760 months from present

$$F_B = 4{,}996\left(\ln\left(\frac{1}{1-0.494}\right)\right)^{1/2.636} =$$

4,319 hours = 164 months from present

$$F_C = 1{,}126 \left(\ln \left(\frac{1}{1 - 0.624}\right)\right)^{1/7.4} =$$

1,122 hours = 36 months from present

1,000 hour inspection is in 800 hours = 32 months from present

Reset Mode A and C to "0"

$$F_A = 96{,}587 \left(\ln \left(\frac{1}{1 - 0.085}\right)\right)^{1/.78} =$$

3,994 hours = 159 months from last inspection

$$F_C = 1{,}126 \left(\ln \left(\frac{1}{1 - 0.194}\right)\right)^{1/7.4} =$$

915 hours = 36 months from last inspection

2,000 hour inspection is 40 months from last inspection

(3) Engine at 500 hours

$$F_A = 96{,}587 \left(\ln \left(\frac{1}{1 - 0.612}\right)\right)^{1/.78} = 89{,}877 \text{ hours} = 3{,}575 \text{ months from present}$$

$$F_B = 4{,}996 \left(\ln \left(\frac{1}{1 - 0.337}\right)\right)^{1/2.858} = 3{,}566 \text{ hours} = 122 \text{ months from present}$$

$$F_C = 1{,}126 \left(\ln \left(\frac{1}{1 - 0.393}\right)\right)^{1/7.4} = 1{,}025 \text{ hours} = 21 \text{ months from present}$$

1,000 hour inspection is in 500 hours = 20 months

Reset Mode A and C to "0"

$$F_A = 96{,}587 \left(\ln \left(\frac{1}{1 - 0.163}\right)\right)^{1/.78} = 9{,}963 \text{ hours} = 398 \text{ months from last inspection}$$

$$F_C = 1{,}126 \left(\ln \left(\frac{1}{1 - 0.774}\right)\right)^{1/7.4} = 1{,}188 \text{ hours} = 47 \text{ months from last inspection}$$

2,000 hour inspection is 40 months from last inspection

(4) Engine at 700 hours

$$F_A = 96{,}587\left(\ell n\left(\frac{1}{1-0.62}\right)\right)^{1/.76} = 92{,}488 \text{ hours} = 3{,}671 \text{ months from present}$$

$$F_B = 4{,}996\left(\ell n\left(\frac{1}{1-0.596}\right)\right)^{1/2.6[illegible]} = 4{,}813 \text{ hours} = 164 \text{ months from present}$$

$$F_C = 1{,}126\left(\ell n\left(\frac{1}{1-0.503}\right)\right)^{1/7.4} = 1{,}072 \text{ hours} = 14 \text{ months from present}$$

1,000 hour inspection is in 300 hours = 12 months

Reset Mode A and C to "0"

$$F_A = 96{,}587\left(\ell n\left(\frac{1}{1-0.857}\right)\right)^{1/.76} = 231{,}776 \text{ hours} = 9{,}270 \text{ months fro[illegible] inspection}$$

$$F_C = 1{,}126\left(\ell n\left(\frac{1}{1-0.794}\right)\right)^{1/7.4} = 1{,}197 \text{ hours} = 47 \text{ months from last inspection}$$

2,000 hour inspection is 40 months from last inspection

(5) Engine at 900 hours

$$F_A = 96{,}587\left(\ell n\left(\frac{1}{1-0.545}\right)\right)^{1/.76} = 70{,}530 \text{ hours} = 2{,}785 \text{ months from present}$$

$$F_B = 4{,}996\left(\ell n\left(\frac{1}{1-0.945}\right)\right)^{1/2.6[illegible]} = 7{,}480 \text{ hours} = 263 \text{ months from present}$$

$$F_C = 1{,}126\left(\ell n\left(\frac{1}{1-0.357}\right)\right)^{1/7.4} = 1{,}008 \text{ hours} = 4 \text{ months from present}$$

1,000 hour inspection is in 100 hours = 4 months

Inspection before failure — F_C = 4.33 months actually

Reset A and C to "0"

$$F_A = 96{,}587\left(\ell n\left(\frac{1}{1-0.429}\right)\right)^{1/.76} = 45{,}077 \text{ hours} = 1{,}803 \text{ months from last inspection}$$

$$F_C = 1{,}126\left(\ell n\left(\frac{1}{1-0.769}\right)\right)^{1/7.4} = 1{,}185 \text{ hours} = 47 \text{ months from last inspection}$$

2,000 hour inspection is 40 months from last inspection

Inspection before failure

A) No Failures

B) 5 engines fail without 1,000 hour inspection
1 additional engine fails before the 2,000 hour inspection

4. CHAPTER 4 ANSWERS

4.1 Problem 4-1

$2 \le \beta \le 5$
Let $\beta_1 = 2$ $\beta_2 = 5$

Consider the case when $\beta_1 = 2$:

$$\eta_1 = \left[\frac{100^2 + 110^2 + 125^2 + 150^2 + 90^2 + 40^2}{2}\right]^{1/2} = 187.0$$

Let T_{now} = Present Time; P_{now} = Prob of Failure at Present

T_{100} = Time in 100 Hours; P_{100} = Prob. of Failure at 100 Hours in Future

Engine	T_{now}	P_{now}	T_{100}	P_{100}
1	40	0.045	140	0.429
2	90	0.207	190	0.644
3	100	0.249	200	0.681
4	110	0.293	210	0.717
5	0	0	100	0.249
6	0	0	100	0.249
		0.794		2.969

Thus, one would expect 2.969 — 0.794 = 2.18 additional failures during the next year.

Consider the case when $\beta_2 = 5$:

$$\eta_2 = \left[\frac{100^5 + 110^5 + 125^5 + 150^5 + 90^5 + 40^5}{2}\right]^{1/5} = 147.3$$

Engine	T_{now}	P_{now}	T_{100}	P_{100}
1	40	0.0015	140	0.540
2	90	0.082	190	0.972
3	100	0.134	200	0.990
4	110	0.207	210	0.997
5	0	0	100	0.134
6	0	0	100	0.134
		0.4245		3.767

Thus, one would expect 3.767 — 0.4245 = 3.34 additional failures during the next year.

So:

As β increases from 2 to 5, the number of expected failures during the next year increases from 2.18 to 3.34.

4.2 Problem 4-2

The predicted design B.1 life = 1000

Assume $\beta = 3$.

Then $\eta = \left[\frac{5(1500)^3 + 5(2000)^3}{1}\right]^{1/3} = 3845.7$

Let L = B.1 life predicted from the Weibayes analysis of the data.

Then:

$$0.001 = 1 - e^{-(L/3845.7)3}$$

$$e^{-(L/3845.7)3} = 0.999$$

$$-(L/3845.7)^3 = \ln 0.999$$

$$\Rightarrow L = 3845.7\ (-\ln 0.999)^{1/3}$$

$$L = 384.6$$

Since the Weibayes analysis predicts B.1 life = 384.6 (which is less than the predicted design value of 1000), one can conclude that the present data is insufficient to increase the predicted design life.

5. CHAPTER 5 ANSWERS

5.1 Problem 5-1 Substantiation Testing

Enter Table 5.1 with a sample size of 20 and β equal to 1.5. The corresponding entry is 0.237. The required test time per bearing is:

$$0.237 \times 3000 \text{ hours} = 711 \text{ hours}.$$

Thus, the zero failures test plan is: run 20 bearings for 711 hours each. If no bearing fatigue failures occur during the test, then the failure mode has been significantly improved, with 90% confidence.

5.2 Problem 5-2

The reliability goal may be stated mathematically as R(2300) = 0.95, which means that the reliability of the vane system is 0.95 (95% succeeding, 5% failing) at 2300 cycles. First, convert this reliability goal to a characteristic life goal: substitute t = 2300 cycles; R(t) = 0.95, and β = 3 into equation 5.2. The results are:

$$\eta = \frac{2300\text{cycles}}{[-\ln(0.95)]}^{1/3}$$

or η = 6190.2 cycles.

The number of test cycles per turbine was not fixed. The only constraint was that it should not exceed 5000 cycles. The table below shows the number of turbines required, assuming 3000, 4000, and 5000 test cycles accumulated on each.

Test Cycles per Turbine	*Ratio of Test Cycles to η = 6190.2*	*Number of Turbines Required — Table 5.2*
3000	0.48	22
4000	0.65	9
5000	0.81	5

Either of these test plans will satisfy the requirements of the test. However, the plan to test 5 turbines for 5000 cycles each requires the fewest total test cycles. (The first plan requires 22 × 3000 = 66,000 cycles, the second plan requires 9 × 4000 = 36,000 cycles, and the third plan requires 5 × 5000 = 25,000 cycles)

Therefore, the test plan that satisfies the test requirements and that requires the fewest total test cycles is: test 5 turbines for 5000 cycles each. If all turbines complete the test, with vane erosion within the allowable limits, then no more than 5% of the turbines will be rejected for excessive erosion prior to 2300 cycles, with 90% confidence.

5.3 Problem 5-3

In the terminology of Section 5.9,

$\beta = 2.5,$
$t = 1000$ hours,
$\eta_o = 2000$ hours,
$\eta_1 = 4000$ hours,
$\alpha_o = 0.1,$
and $\alpha_1 = 0.9.$

Equations 5.3 and 5.4 will now be solved for r_o and n, using the method of Section 5.9.

Step 1: $p_o = 1 - \exp(-(1000/2000)^{2.5}) = 0.162$
$p_1 = 1 - \exp(-(1000/4000)^{2.5}) = 0.031$

Step 2: Setting $r_0 = 0$, n_o, the value of n satisfying equation 5.3, was found to be 14. n_1, the value of n satisfying equation 5.4, is 3.

$$a = \frac{14 \times 0.162}{3 \times 0.031} = 24.4$$

$$b = \frac{0.162}{0.031} = 5.23$$

Step 3: Since, for $r_o = 0$, a is greater than b, r_o is increased by 1.

For $r_o = 1$, $n_o = 23$ and $n_1 = 18$, giving $a = 6.677$.

Step 4: a is still greater than b, for $r_o = 1$. For $r_o = 2$, $n_o = 31$ and $n_1 = 36$, giving $a = 4.5$.

Step 5: For $r_o = 2$, a is less than b, so the process of increasing r_0 is stopped here. The a-ratio for $r_o = 2$ is 4.5 and is closer to $b = 5.23$ than the a-ratio for $r_o = 1$. So, the final value of r_o is 2.

Step 6: The final value of n is

$$n = \frac{n_o + n_1}{2}$$

$$= \frac{31 + 36}{2}, \text{for } r_o = 2$$

$$n = 33.5$$

$$\text{or } n = 33 \text{ (rounding 33.5)}$$

The final test plan is: test 33 units for 1000 hours. If 2 or fewer units fail while on test, the test is passed. Note that the additional requirement in this plan has more than doubled the number of units and test time required. (The zero-failures test plan required that 14 units be tested 1000 hours each.)

6. CHAPTER 6 ANSWERS: None

7. CHAPTER 7 ANSWERS

7.1 Problem 7-1

Using equation 7.1,

$$1.5e^{(\ 0.78(1.645)/40)} \leq \beta \leq 1.5\, e^{(0.78(\ 1.645)/40)}$$

$1.22 \leq \beta \leq 1.84$ is the 90% confidence interval for β.

Using equation 7.2,

$$2000e^{(\ 1.051.645)/1.5\ 40)} \leq \eta \leq 2000\, e^{(1.0\ 5(1.645)/1.5\ 40)}$$

$1667 \leq \eta \leq 2399$ is the 90% confidence interval for η.

7.2 Problem 7.2

Using equation 7.3a, b, c, and d.

$$7.3a\ U = (\ln(1500) - \ln(2000)) \cdot 1.5 = -0.4315$$

$$7.3b\ Var\,(U) = [\ 1.168 = (-0.4315)^2\,(1.1) - (0.1913)\,(-0.4315)\ ]\ \frac{1}{40} = 0.0364$$

$$7.3c\ U_1 = (-0.4315) - (1.645)\,(0.0364)_{1/2} = -0.7454$$

$$U_2 = (-0.4315) + (1.645)\,(0.0364)^{1/2} = -0.1177$$

$$7.3d\ e^{-e^{(-0.1177)}} \leq R(1500) \leq e^{(-e^{(-0.7454)})}$$

Therefore, $0.411 \leq R(1500) \leq 0.622$

7.3 Problem 7.3

For n = 40, from Appendix Tables II.2 and II.3

$F_{1,0.05}$	= 0.001	$F_{1,0.95}$	= 0.072
$F_{2,0.05}$	= 0.008	$F_{2,0.95}$	= 0.113
$F_{3,0.05}$	= 0.020	$F_{3,0.95}$	= 0.149

$\beta = 1.5 \quad \eta = 2000$

Using equation 7.4

$$t_{1,0.05} = 2000\left[\ln\frac{1}{1-0.001}\right]^{1/1.5} = 20.01$$

$$t_{2,0.05} = 2000\left[\ln\frac{1}{1-0.008}\right]^{1/1.5} = 80.21$$

$$t_{3,0.05} = 2000\left[\ln\frac{1}{1-0.020}\right]^{1/1.5} = 148.36$$

$$t_{1,0.95} = 2000\left[\ln\frac{1}{1-0.072}\right]^{1/1.5} = 354.81$$

$$t_{2,0.95} = 2000\left[\ln\frac{1}{1-0.113}\right]^{1/1.5} = 486.33$$

$$t_{3,0.95} = 2000\left[\ln\frac{1}{1-0.149}\right]^{1/1.5} = 592.74\ 1$$

90% confidence intervals on first 3 failures:

Failure 1: 20.01 ≤ Time ≤ 354.81

Failure 2: 80.21 ≤ Time ≤ 486.33

Failure 2: 148.36 ≤ Time ≤ 592.74.

7.4 Problem 7.4

Use Weibull-Thorndike Chart

Figure 7.6 (β = 1)

T/η = 4000/1000 = 4.

We want 0.9 probability bands on the number of failures occurring at T = 4000 hours.

Entering x-axis at T/η = 4,

a. **When p = 0.05, C = 0**

b. **When p = 0.95, C = 7.**

Thus, a 0.9 probability band on number of failures by T = 4000 hours is (0,7).

7.5 Problem 7.5

Steps 1 and 2: Completed on Figure J.8

β = 1.59 η = 258.0

Step 3: MTTF = 1/10 (51 + . . . + 451) = 1/10 (2261) = 226.1

Step 4: On graph.

Step 5: 10/2000 × 100 = 0.5% of the population failed

Step 6: On graph Figure J.9

η new ≃ 6400 hours.

7.6 Problem 7.6

Use the technique of Section 7.6 to calculate bands on Weibull from Figure J.10

n = 10 k(n) = 0.246

Using $F(x) = 1 - e^{(-x/\eta)\beta}$, Bands are:

$$[F(x) - 0.246, F(x) + 0.246]$$

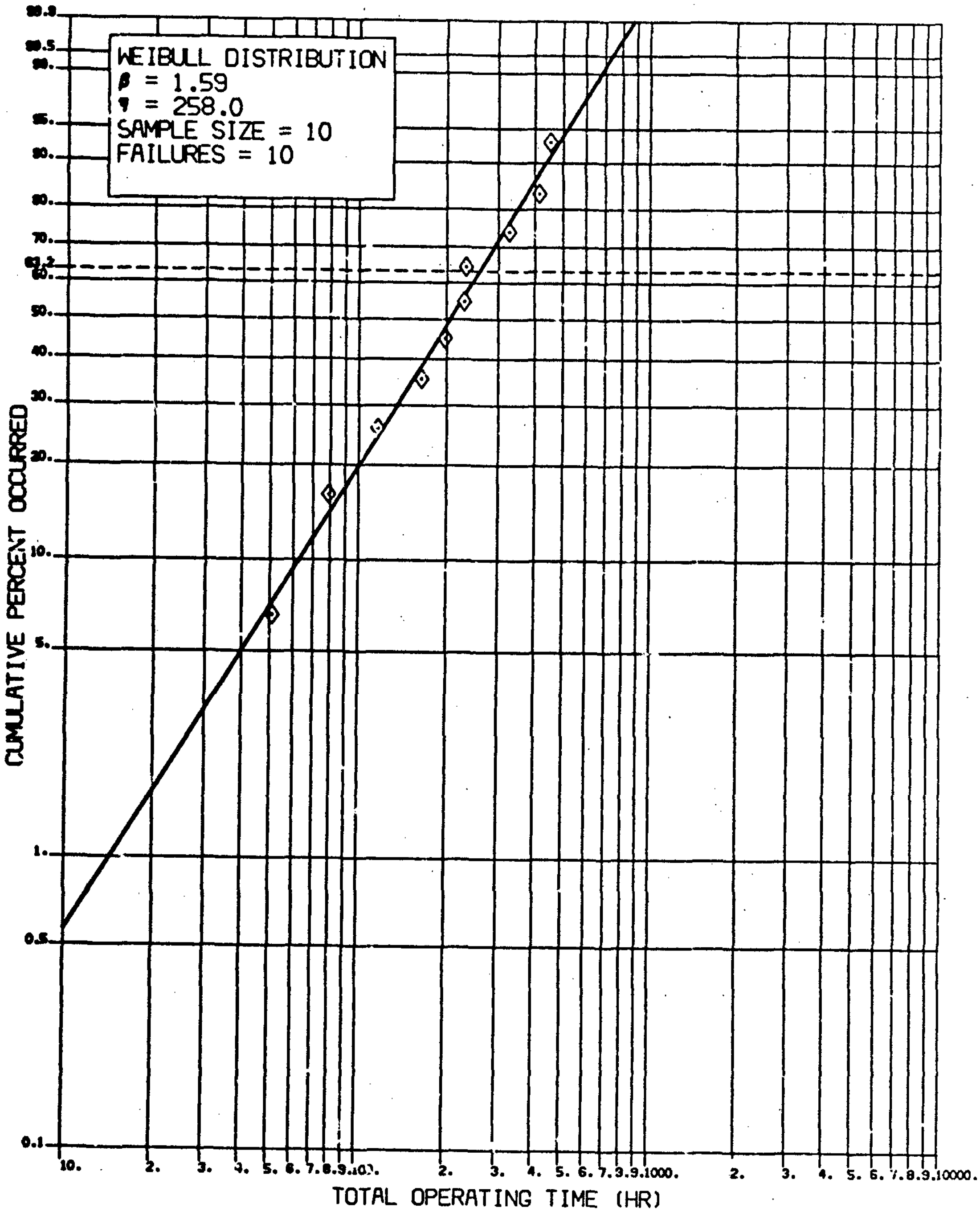
For

$\beta = 2.974 \quad \eta = 895.42$

$$1 - e^{-[x/895.42]^{2.974}} - 0.246 \leq 1 - e^{-[x/895.42]^{2.974}} + 0.246$$

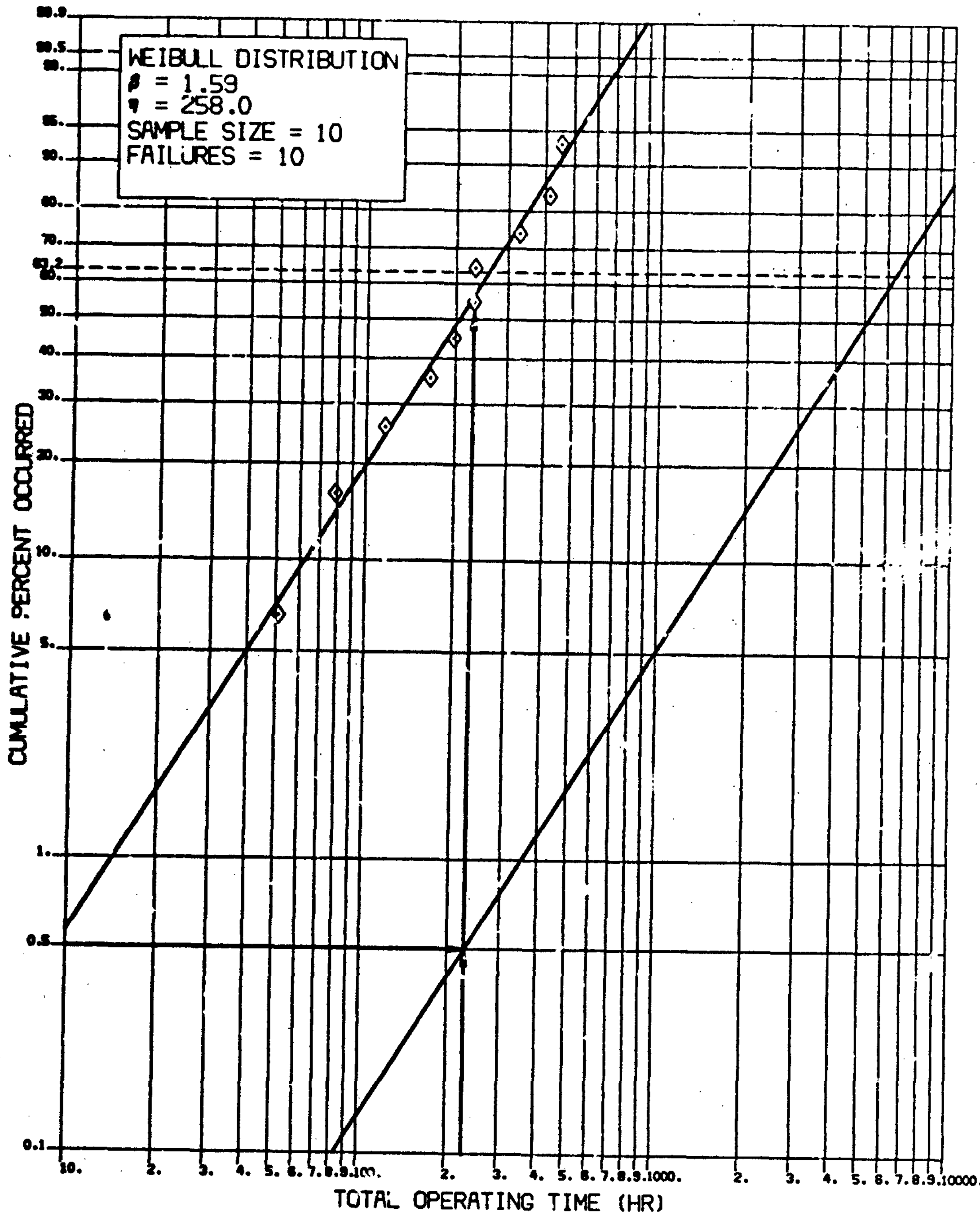
x	*Upper Band*	*Lower Band*
100	0.247	0
200	0.253	0
300	0.284	0
400	0.333	0
500	0.408	0
600	0.508	0.016
800	0.757	0.265
1000	0.997	0.505
1200	1.0	0.662
2000	1.0	0.754

Now, plot on Figure J.11; since "true" Weibull lies partly outside the confidence bands, we must conclude that the Weibulls are significantly different.



FD 272256

Figure J.8. Problem 7-5



FD 272255

Figure J.9. Problem 7-5

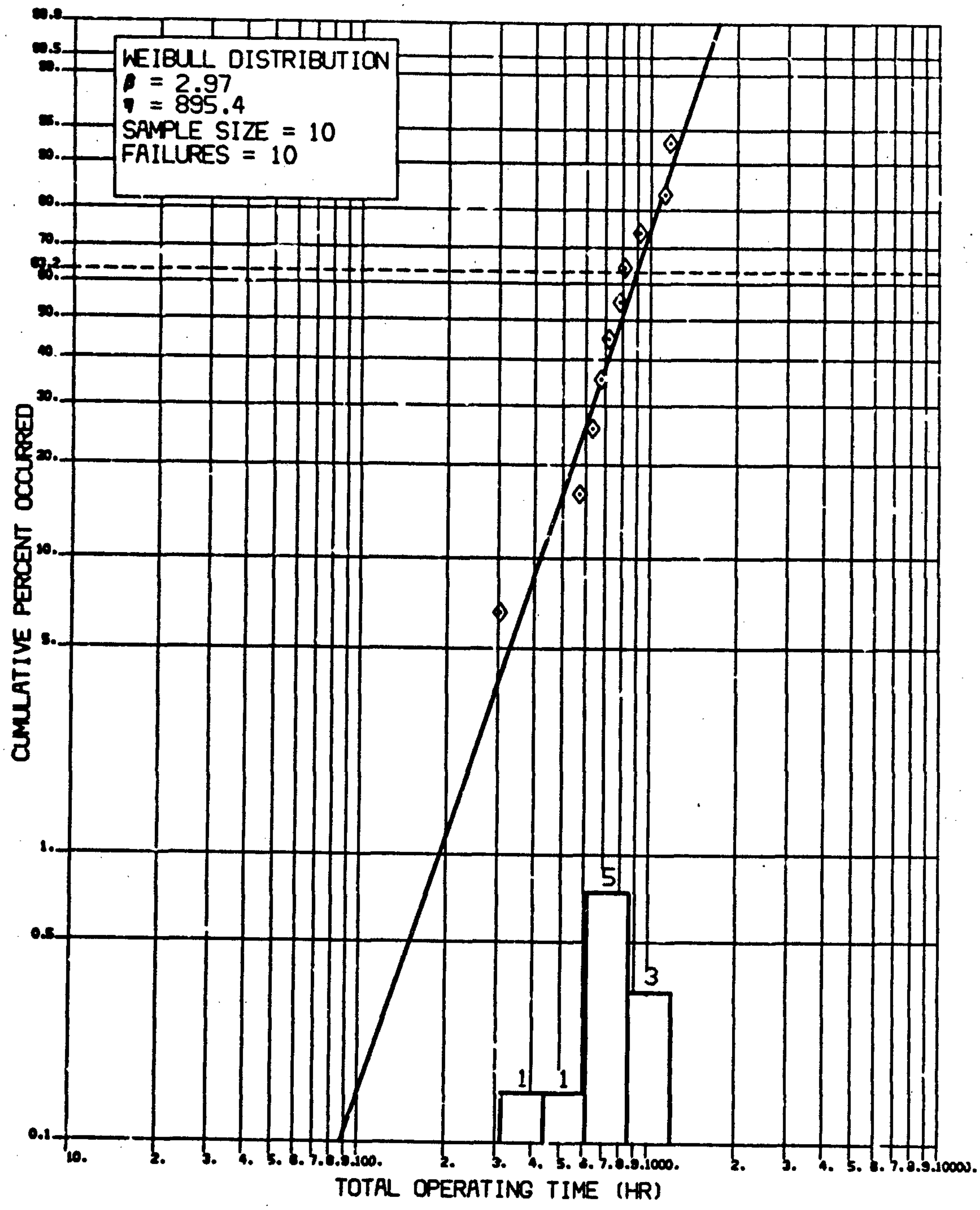
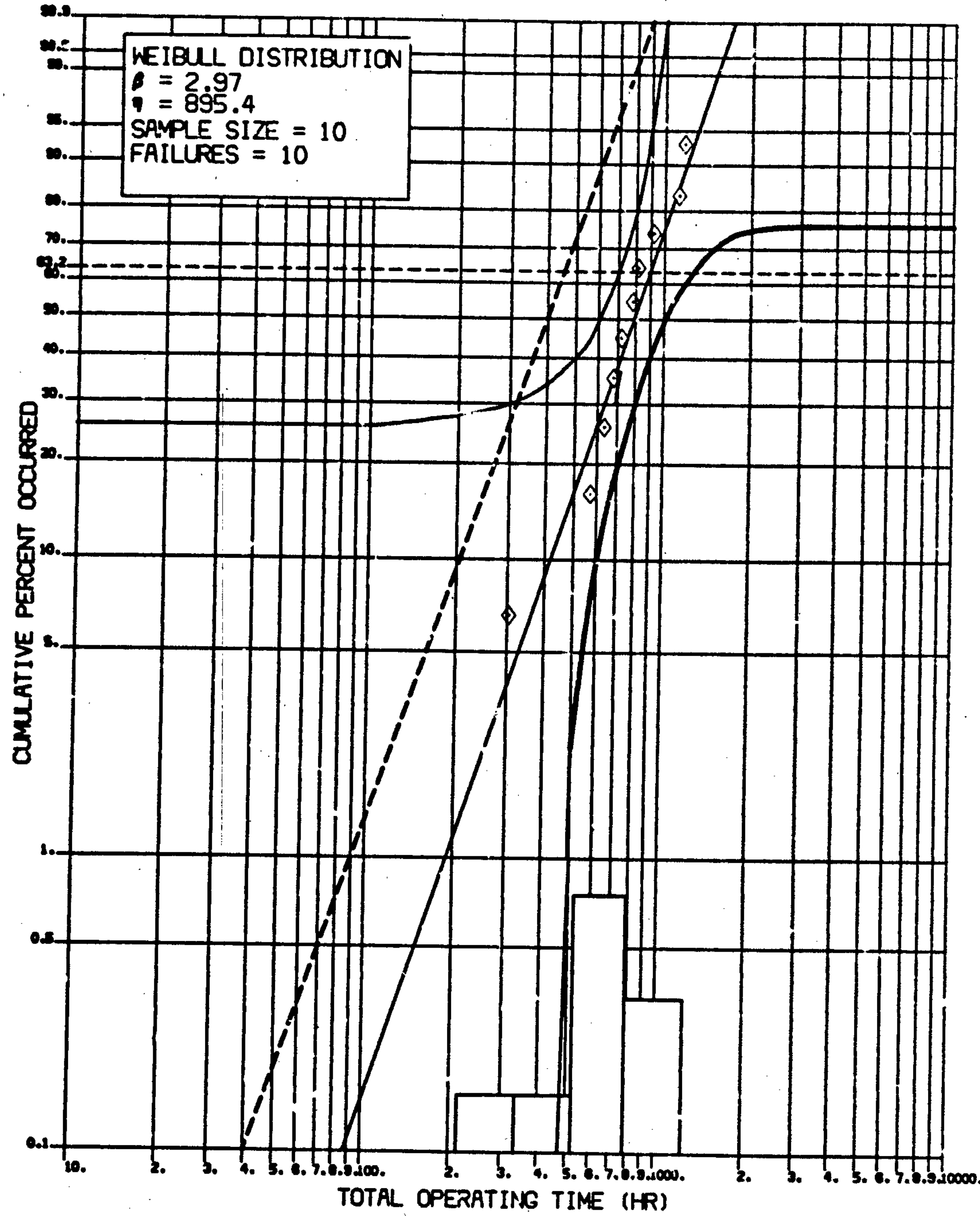


Figure J.10. Problem 7-6



FD 272258

Figure J.11. Problem 7-6

*U.S.Government Printing Office: 1984 — 759-062/918